Numerical Experiments on Algebraic Properties of Scale-Free and Small-World Networks

Masahiro Mizutani

Contents

1 Introduction 1

2 A little matrix theory of graph 2
2.1 Graph 2
2.2 Adjacency matrix 4
2.3 Laplacian matrix 8

3 Small-world and Scale-free networks 10
3.1 Small-world network 10
3.2 Scale-free network 13
3.3 Spectral densities 15

4 Numerical experiments 19
4.1 Watts and Strogatz model 19
4.2 Barabási and Albert model 25

5 Conclusion 31

1 Introduction

Discoveries of the constructive model of the samll-world network by Watts and Strogatz[22] in 1998, the real network with the scale-free network by Albert, Jeong and Barabási[2] in 1999 and the generative model of the scale-free network by Barabási and Albert[4] opened a new frontier in the network science, and their influences range

from physics, mathematics to biology and sociology[3][10][19][18][20]. Their models proposed very simple procedures generating a network with the small-world and the scale-free properties respectively. These specific properties are not visible by considering just only on each entry building the network but by observing the networks as a whole statistically, therefore need the giant complex networks.

In this paper we examine the algebraic properties of rather small networks generated by the Watts and Strogatz model(Sec.3.1) and the Barabási and Albert model(Sec.3.2). Actually one of the main problems of algebraic graph theory is to analize algebraic structure of the networks for which their topological properties must be reflected[23][9][7]. The above pioneer works stimulate extensive studies for the statistics on the algebraic structures, in numerically[13][15] and in theoretically[12][17]. However in cases of small size networks where the asymptotic statistics or the theory of random graph are not applicable, the properties of this kind of networks are still less well known.

2 A little matrix theory of graph

The adjacency and the Laplacian matrices are both naturally associated with a graph. The relationship between the algebraic properties of these matrices and the usual properties of the graph have been studied extensively[7][8][9][23], and it is well known that in general the connectivity cannot be determined from the eigenvalues alone, and also from the eigenvalue of a graph, the diameter and the degree sequence of a graph cannot be determined. We summarize below some of the known facts in the matrix theory concerning our numerical experiments.

2.1 Graph

A *graph* G is a pair of a sets (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of *vertices* in the graph and $E = \{e_1, e_2, \ldots, e_m\}$ a set of pairs of vertices called *edges*. The number of vertices $n = |V|$ in a graph is called an *order* of the graph, sometimes simply denoted by $|G|$, and the number of edges $s = |E|$ a *size* of the graph.

A graph with edges without orientations, which we mainly consider in this paper, is called an *undirected graph* or simply a graph, where an edge between vertices u and w is denoted by an unordered pair $\{v, w\}$ or simply vw, saying that this edge joins v and w and that it is *incident* with v and w. Two edges are adjacent if they have a vertex in common. In this case, vertices v and w are *adjacent* each other. A graph with edges with orientations is called a *digraph*, where an edge joining vertices from v

to w is denoted by a ordered pair (v, w) or simply vw, giving the edge an orientation $\overrightarrow{vw}$. In this case, the vertex v is adjacent to w, but w not adjacent to v.

Edges joining vertices to themselves are called *loops*. A *multigraph* is the graph with multiple edges, meaning two vertices are joined by more than one edge, while a *simple graph* is a graph with no loops where each edge has multiplicity at most one. The set of vertices adjacent to a vertex v in G, the neighbourhood of v, is denoted by $\Gamma_G(v)$. The number of the neighbourhood $\Gamma_G(v)$ of v is called its *degree* or *valency*, denoted by $\deg_G(v_i) = |\Gamma_G(v)|$. A graph G is *k-regular*, if every vertex of G has the same degree equal to k.

A *walk* in a graph is an alternating sequence of vertices and edges, say $v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_m}, v_{i_m}$, in which each edge $e_{i_k} = v_{i_{k-1}} v_{i_k}$. The walk going from v_{i_0} to v_{i_m} is often shortened to $v_{i_0} v_{i_1} \ldots v_{i_m}$. Its *length* is m, the number of occurrences of edges in v_{i_0}-v_{i_m} walk. A walk in which no vertex is repeated is called a *path*. A *cycle* of length ℓ in a graph is a walk $(v_{i_0}, v_{i_1}, \ldots, v_{i_\ell})$ with $v_{i_0} = v_{i_\ell}$, $\ell \neq 0$ and $v_{i_1}, \ldots, v_{i_\ell}$ all distinct. The *distance* between two vertices v and w in G is defined by the minimal length of a path from v to w, denoted by $d_G(v, w)$. The *mean path length* or *characteristic path length* is an average of $d_G(v, w)$ over any pair of vertices v and w in G. Then the maximum distance between two any vertices in a graph $\operatorname{diam} G = \max_{v,w \in G} d_G(v, w)$ is called the *diameter* of the graph G. A graph (V, E) is *connected* if $V \neq \emptyset$ and there exists a walk between any two distinct vertices of V. A *tree* is a connected graph without cycles.

If G and H are graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$, the H is a *subgraph* of G. H is an *induced subgraph* of G if $E(H)$ contains all edges from $E(G)$ with endpoints in $V(H)$. A *spanning tree* of a connected graph (V, E) is a subgraph (V', E') with $V' = V$, which is a tree.

The *clustering coefficient* $C(G)$ or the *network transitivity* of a graph G is important in the theory of social networks[21] and defined by

$$C(G) = \frac{\text{number of pairs of vertices } ab, ac \text{ of adjacent edges for which } bc \text{ is an edge}}{\text{number of pairs } ab, ac \text{ of adjacent edges}}$$
$$= \frac{6 \times \text{number of triangles in } G}{\text{number of paths of length two in } G},$$

where a path refers to a distinct path starting from a specifies vertex[18][11].

2.2 Adjacency matrix

The adjacency matrix $A_G = (A_{ij})$ of the graph G with vertices $\{v_1, v_2, \ldots, v_n\}$ is $n \times n$ matrix, whose (i, j) entry is defined by

$$A_{ij} = \text{the number of edges beginning at } v_i \text{ and ending at } v_j.$$

Since there exist an one to one correspondence between the graph G and its adjacency matrix A, therefore studying properties of the graph G is in principle reduced to consider its adjacency matrix A_G. Different graphs on the same vertex set have different adjacency matrices, even if they are isomoriphic. If graphs G_1 and G_2 are isomorphic, then their adjacency matrices A_{G_1} and A_{G_2} are similar and we identify G_1 with G_2 in a following sense.

Lemma 1 ([14]). *Let G_1 and G_2 be graphs on the same vertex set. Then they are isomorphic if and only if there is a permutation matrix P such that $P^T A_{G_1} P = A_{G_2}$.*

The characteristic polynomial of a matrix A_G is the polynomial

$$\phi(A_G, x) = \det(xI - A_G),$$

and let $\phi(G, x)$ denote the characteristic polynomial of A_G, where I is the identity matrix with order n. The spectrum of a matrix is the list of its eigenvalue together with their multiplicities. Lemma 1 shows that $\phi(G_1, x) = \phi(G_2, x)$ if G_1 and G_2 are isomorphic, and the spectrum is an invariant of the isomorphism class of a graph.

The *eigenvalues of a graph* G are the eigenvalues of its adjacency matrix A_G. The *spectrum of a graph* G is a set of eigenvalues with their multiplicities. Two graphs are *cospectral* whenever they have the same spectrum. So a graph G is in general not determined by its spectrum alone, but determined by its spectrum only if every graph cospectral with G is isomorphic to G. Table 1 shows several spectrums of special graphs.

graph type	eigenvalues	respective multiplicities
complete K_n	$n-1, -1$	$1, n-1$
complete bipartite $K_{r,s}$	$\sqrt{rs}, 0, -\sqrt{rs}$	$1, r+s-2, 1$
cycle C_n	$2\cos\left(\frac{2k\pi}{n}\right), k = 1, \ldots, n$	$\begin{cases} 2, 1, \ldots, 1, 2 \text{ for } n \text{ even} \\ 1, 1, \ldots, 1, 2 \text{ for } n \text{ odd} \end{cases}$
path P_n	$2\cos\left(\frac{k\pi}{n+1}\right), k = 1, \ldots, n$	$1, 1, \ldots, 1$

Table. 1 Spectrums of graphs K_n, $K_{r,s}$, C_n and P_n.

The spectral radius $\rho(A)$ of a matrix A is the maximum of the moduli of its eigenvalues. Remark that the spectral radius need not be an eigenvalue of it. The entries A_{ij} of the adjacency matrix, are all non-negative, so we can apply the Perron-Frobenius theorem to the adjacency matrix A_G.

Theorem 2 (Perron-Frobenius). *Suppose A is a real irreducible non-negative $n \times n$ matrix. Then we have a), b) and c) below:*

a) $\rho(A)$ is a maximal eigenvalues λ_{max} of A, and all entries of its eigenvector for α_{max} are non-zero with same sign.
b) α_{max} is simple, i.e., multiplicity 1.
c) For any non-negative $n \times n$ matrix B such that $A - B$ is non-nnegative, $\rho(B) \leq \rho(A)$, with equality if and only if $B = A$.

The type of the adjacency matrix A_G depends on different variation of the graph G. A_G is symmetric in case of the unordered graph, while possibly asymmetric in case of the digraph. If G is a multigraph, then A_G is a non-negative integer matrix with zero on the diagonal. If loops are permitted, then A_G is a general non-negative integer matrix. The simple graph, meaning that it is loop-free and a pair of vertices are joined by at most one edge, has a (0,1)-matrix A_G with zero on the diagonal. If the graph is unordered and simple, then A_G is a symmetric (0,1)-matrix with zero on the diagonal. We consider hereafter unordered and possibly simple graphs.

We have a following corollary for an undirected graph or a symmetric non-negative matrix A_G, which is a direct consequence of the Perron-Frobenius theorem.

Corollary 3. *Let G be a undirected graph, then then A_G has n real eigenvalues $\alpha_{max} = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n = \alpha_{min}$ where α_1 is simple and $|\alpha_i| \leq \alpha_1$ for $i = 2 \ldots, n$, and a corresponding orthnormal set of eigenvectors by which A_G is diagonalized to* $\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Next theorem indicate a relation focused on eigenvalues between a graph and its induced subgraph.

Theorem 4 (Interlacing theorem[7][14]). *Let H is an induced subgraph of G. If $\eta_1 \geq \eta_2 \ldots \geq \eta_\ell$ are the eigenvalues of H, and if $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ are the eigenvalues of G, then*

$$\alpha_{i+n-\ell} \leq \eta_i \leq \alpha_i, \text{ for } i = 1, 2, \ldots, \ell.$$

The eigenvalues of the symmetric matrix A have a relation to its *Rayleigh quotient* $R(A; \mathrm{x})$, which is defined using the inner product for any non-zero vector $\boldsymbol{x}$ as

$$R(a; \boldsymbol{x}) = \frac{(\boldsymbol{x}, A\boldsymbol{x})}{(\boldsymbol{x}, \boldsymbol{x})}.$$

Indeed, for the adjacency matrix A_G of the undirected graph G the Rayleigh quotient has a value within the range from the smallest to the largest eigenvalues.

Theorem 5. *Let G be an undirected grapg with adjacency matrix A_G and with eigenvalues $\alpha_{max} = \alpha_1 \geq \alpha_2 \geq \ldots \alpha_n = \alpha_{min}$. Then $R(A_G, \boldsymbol{x})$ takes on precisely the values in the interval $[\alpha_1, \alpha_n]$. In addition, $R(A_G, \boldsymbol{x}) = \alpha_{max}$ or α_{min} only if $\boldsymbol{x}$ is a corresponding eigenvector.*

Corollary 6. *If H is a proper induced subgraph of c connected undirected graph G, if H has maximum eigenvalue η_{max} and minimum eigenvalue η_{min}, if G has maximum eigenvalue α_{max} and minimum eigenvalue α_{min}, then $\alpha_{min} \leq \eta_{min} \leq \eta_{max} \leq \alpha_{max}$.*

Corollary 7. *If e is an edge of a connected undirected graph G, then the largest eigenvalue of G is strictly greater than the largest eigenvalue of $G \backslash e$.*

By considering the values of entries $(A\boldsymbol{x})_i / \boldsymbol{x}_i (i = 1, 2, \ldots n)$ for a vector with non-negative entries, the largest eigenvalue α_{max} of A satisfies

$$\min_i \left\{ \frac{(A\boldsymbol{x})_i}{\boldsymbol{x}_i} \right\} \leq \alpha_{max} \leq \max_i \left\{ \frac{(A\boldsymbol{x})_i}{\boldsymbol{x}_i} \right\}.$$

Then taking the all-1 vector $\boldsymbol{j} = (1, 1, \ldots, 1)^t$ as above $\boldsymbol{x}$ and A_G for an undirected graph G with degrees $d_1, d_2, \ldots, d_n$, we obtain

$$\min_i d_i \leq \alpha_{max} \leq \max_i d_i.$$

Thus using Theorem 5 with the vector $\boldsymbol{j}$, this results the following theorem saying that the maximum eigenvalue lies between the *average degree* $\delta(G) = \frac{1}{n} \sum_{i=1}^{n} d_i$ and the *maximum degree* $\Delta(G) = \max_i d_i$ of the graph G, and equals both value iff the graph is regular.

Theorem 8. *If G is an undirected graph with degrees $d_1, d_2, \ldots, d_n$ and maximum eigenvalue α_{max}, then*

$$\delta(G) \leq \alpha_{max} \leq \Delta(G).$$

Equality is attained if and only if the graph is regular.

Moreover we have

Theorem 9 ([23]). *Let $G = (V, E)$ be a graph with order $n = |V|$ and size $s = |E|$ and maximum eigenvalue α_{max}. Then*

(1)

$$\frac{2s}{n} \leq \alpha_{max} \leq \sqrt{\frac{2s(s-1)}{n}}.$$

The left equality is attained if and only if the graph is regular and the right quality attained if and only if the graph is the perfect graph K_n.

(2)

$$\frac{1}{s} \sum_{all\ i,j} \sqrt{d_i d_j} \leq \alpha_{max} \leq \max_i \frac{1}{d_i} \sum_{\substack{j \\ A_{ij} \neq 0}} \sqrt{d_i d_j}$$

The most fundamental relationship between the eigenvalues of a graph and its geometrical properties concerns walks.

Theorem 10. *Suppose A_G is the adjacency matrix of G, then for $k = 0, 1, \ldots)$ (i,j)-entry A^k_{Gij} of $A^k_G = \underbrace{A_G \circ A_G \circ \ldots \circ A_G}_{k times}$ is the number of v_i-v_j walks of length k.*

Theorem 11 ([7]). *If a graph G has the diameter $\mathrm{diam}(G)$ and t distinct eigenvalues, then*

$$\mathrm{diam}(G) \leq t - 1.$$

Thus we have a following elementary theorem.

Theorem 12 ([7]). *Let A_G be the adjacency matrix of a graph G with degree sequence $\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n)$. Then we have*

(a) $\deg_G(v_i) = A^2_{Gii}$;

(b) the sum of the degrees of G is even, and exactly twice the number of edges

$$\sum_{i=1}^{n} \deg_G(v_i) = 2|E(G)| = \mathrm{tr}A_G^2;$$

(c) the number of triangles in G is $\frac{1}{6}\mathrm{tr}A_G^3$;
(d) the number of 4-cycles in G is $\frac{1}{8}\left(\mathrm{tr}A_G^4 + \mathrm{tr}A_G^2 - 2\sum_{i=1}^{n} A_{Gii}^2 A_{Gii}^2\right)$;
(e) the number of 5-cyles in G is $\frac{1}{10}\left(\mathrm{tr}A_G^5 + 5\mathrm{tr}A_G^3 - 5\sum_{i=1}^{n} A_{Gii}^2 A_{Gii}^3\right)$.

Corollary 13. *The clustering coefficient $C(G)$ defined in Sec.2.1 is given by*

$$C(G) = \frac{\mathrm{tr}A_G^3}{\sum_{i\neq j} A_{Gij}^2}.$$

2.3 Laplacian matrix

Let G be a simple graph with adjacency matrix A_G. Suppose D_G be $\mathrm{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$, the diagonal matrix with the degrees of G on the diagonal (with the same vertex ordering as in A_G). Then the *Laplacian matrix* L_G or simply Laplacian of G is defined by

$$L_G = D_G - A_G.$$

The *Laplacian eigenvalues* of a simple graph G are the eigenvalues of the Laplacian matrix L_G. Define the $n \times s$ *oriented incidence matrix* Q_G with rows and colums indexed by V and E of $G = (V, E)$ whose adjacency matrix is A, whre $n = |V|$ and $s = |E|$, by

$$(Q_G)_{v,e} = \begin{cases} A_{vw} & \text{if } v \text{ is the initial vertex of } e = vw, \\ -A_{vw} & \text{if } v \text{ is the terminal vertex of } e = vw, \\ 0 & v \text{ and } e \text{ are not incident or e is loop.} \end{cases}$$

Then we can directly verify the following thorem.

Theorem 14. *Let G be a graph, and let Q_G be an oriented incident matrix with respect to the orientation of its edges. Then*

$$L_G = Q_G Q_G^t.$$

Theorem 14 shows taht the Laplacian matrix L_G of the graph is independent of the orientation of the graph.

The *complexity of graph* or *tree number* is defined by a number of the spanning trees of G. For any matrix M with rows and columns indexed by the set $\Omega = \{\omega\}$, let $M[\omega]$ denote the submatrix of M obtained by deleting the row and column indexed by element ω of Ω.

Theorem 15 (Matrix-tree threorem[8][7][9][14][23]). *Let G be a graph and let L_G be its Laplacian matrix. If v is an arbitrary vertex of G, then $\det L_G[v]$ is common and equal to the complexity of the graph, i.e., the number of spanning trees of G.*

Corollary 16 ([14][23]). *The number of spanning trees of K_n is n^{n-2}.*

The matric L_G acts on $\mathbb{R}^V$ as a linear operator, where $\mathbb{R}^V = \{\boldsymbol{f} : V \to \mathbb{R}\}$ be the set of functions from V to $\mathbb{R}$, denote by $\boldsymbol{f}_v$ the value of $\boldsymbol{f}$ at the vertex v. Then $(\boldsymbol{f}, L_G\boldsymbol{f})$ is a natural quadratic form associated with L_G.

Theorem 17. *Let A_G be an adjacency matrix and L_G be a Laplacian matrix of G. Then*

$$(\boldsymbol{f}, L_G\boldsymbol{f}) = \sum_{vw \in E} A_{Gvw}(\boldsymbol{f}_v - \boldsymbol{f}_w)^2.$$

By the definition L_G is a real symmetric, and Theorem 17 implies it is positive semi-definite. Therefore there exist n non-negative real eigenvalues of L_G, called the *Laplace eigenvalues* of the graph G, denoting them in non-decreasing order

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = \lambda_{max},$$

where λ_i is the ith smallest eigenvalue and remark that $0 = \lambda_1$ is always an eigenvalue of L_G and $\boldsymbol{j} = (1, 1, \ldots, 1)^t$ is a corresponding eigenvector.

The next theorem says that the graph G is connected if and only if $\lambda_1 = 0$ is simple.

Theorem 18 ([7]). *The multiplicity of 0 as an eigenvalue of L_G is equal to the number of connected components of G.*

Using the Rayleigh quotient $R(L_G, \boldsymbol{x})$, the kth smallest eigenvalue λ_k is characterized by the Courant-Fisher min-max formula[7]

$$\lambda_k = \min_U \max_{\boldsymbol{x}} \{R(L_G, \boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^V, (\boldsymbol{x}, \boldsymbol{x}) = 1\},$$

where the minimum is taken over all k-dimensional subspace U of $\mathbb{R}^V$, or

$$\lambda_k = \min\{R(L_G, \boldsymbol{x}) : (\boldsymbol{x}, \boldsymbol{x}) = 1, \boldsymbol{x} \perp \boldsymbol{x}^i, 1 \leq i \leq k\},$$

where $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^{k-1}$ are pairwise orthogonal eigenvectors of $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$, espacially

$$\lambda_2 = \min\{R(L_G, \boldsymbol{x}) : (\boldsymbol{x}, \boldsymbol{x}) = 1, \boldsymbol{x} \perp \boldsymbol{j}\},$$

which yields the following results[7].

Lemma 19. *For any non-adjacent vertices q and r in a graph G,*

$$\lambda_2 \leq \frac{1}{2}\left(\deg_G(q) + \deg_G(r)\right).$$

Theorem 20. *For any graph $G = (V, E)$ of order $n = |V|$,*

$$\lambda_2 \leq \frac{n}{n-1}\delta(G) \qquad \textit{and} \qquad \frac{n}{n-1}\Delta(G) \leq \lambda_{max} \leq 2\Delta(G),$$

where $\delta(G)$ is the average degree and $\Delta(G)$ the maximum degree of G. If G is simple, the last inequality can be strengthened to $\lambda_{max} \leq \max\{\deg_G(v) + \deg_G(w) : vw \in E\}$.

We have an another version of the Matrix-tree Theorem(Theorem 15).

Theorem 21 ([14]). *Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of the Laplacian L_G of G. Thus the number of spanning tree in G is $\frac{1}{n}\prod_{i=2}^{n}\lambda_i$.*

3 Small-world and Scale-free networks

The network is considered mathematically as the graph. We summarize here the basic properties of the small-world and the scale-free networks.

3.1 Small-world network

The small-world networks means casually that there is a relatively short path between any two vertices in despite of their huge order of the graph. This small-world effect often characterizes the complex networks[1][18]. To define this precisely, consider a connected undirected network for sake of simplicity. The mean path length $\langle \mathrm{d}_G \rangle$ between vertex pairs in the graph G is given by

$$\langle \mathrm{d}_G \rangle = \frac{1}{\frac{1}{2}n(n-1)} \sum_{i \geq j} d(v_i, v_j),$$

where $d(v_i, v_j)$ is the distance from vertex v_i to vertex v_j. Then we say that the networks has the *small-world property* if the value of $\langle \mathrm{d}_G \rangle$ scales logarithmically or

slower with the order n of the graph with fixed mean degree, that is,

$$\langle \mathrm{d}_G \rangle \sim \log n.$$

This logarithmic scaling of the mean path length can be proved for a variety of network models, including the Erdös-Rényi random graphs[9] which are however not scale-free.

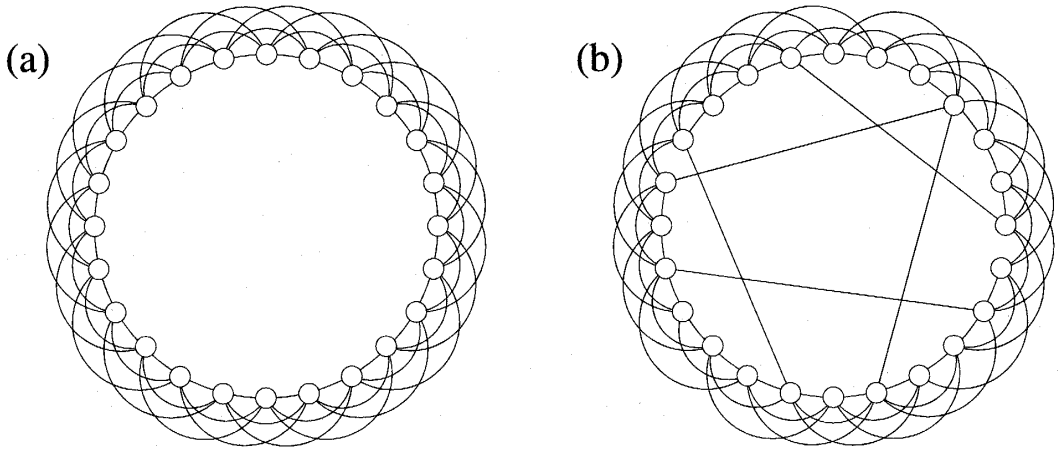

Fig. 1 From [18, Fig.11]. (a) A symmetric k-regular graph(k is even) with edges between all vertex pairs separated by $k/2$ or fewer distance spacing, with $k = 6$ in this case. (b) The small-world model[22] is created by choosing at random a fraction p of the edges in the graph and moving one end of each to a new location, also chosen uniformly at random.

The small-world network attracted widely attention when Watts and Strogatz showed by the computer simulation that a network which originally does not have the samll-world effect can drastically become to be small-world even if very slight randomness modifies the network[22]. They proposed a one-parameter model constructing the small-world network changing from a completely ordered finite dimensional lattice to a fully random graph, starting from the symmmetric k-regular graph($k \geq 4$ is even) with edges between all vertex pairs separated by $k/2$ or fewer distance spacing. Fig.1(a) indicates this symmetric 6-regular graph with 24 vertices is realized on a ring in which every vertex is joined to its first 3 neighbor on either side.

The algorithm constructing the small-world network, which hereafter we call the Watts and Strogatz(WS) model, is the following[22][1, VI.A][18, VI]:

1. Start with order: Start with the symmetric k-regular graph with n vertices, which has $kn/2$ edges(Fig.1(a)). Suppose $N \gg k \log n \gg 1$ to consider a

connected sparse graph.

2. Randomly rewiring: With probability p by randomly taking small fraction of the edges in the graph rewire those $pkn/2$ edges as shortcuts such that
   2-1 moving randomly chosen one end of each taken edge
   2-2 to a new vertex chosen uniformly at random from the lattice(Fig.1(b)).

A graph $WS_p(n,k)$ created by above procedure allows to interpolate between a regular lattice and a random graph. In fact, when $p = 0$, the mean path length $\langle \mathrm{d}_{WS_0(n,k)} \rangle$ tends to $n/2k$ for large n, half of the diameter $\mathrm{diam}WS_0(n,k)$, showing no small-world property. When $p = 1$, the rewired graph becomes almost a random graph, with typical mean path length on the oder of $\log n / \log k$, showing the small-world property. Watts and Strogatz discovered by numerical simulation that there is a p-region in which clustering is high and mean distance simultaneously low as shown in Fig.2.

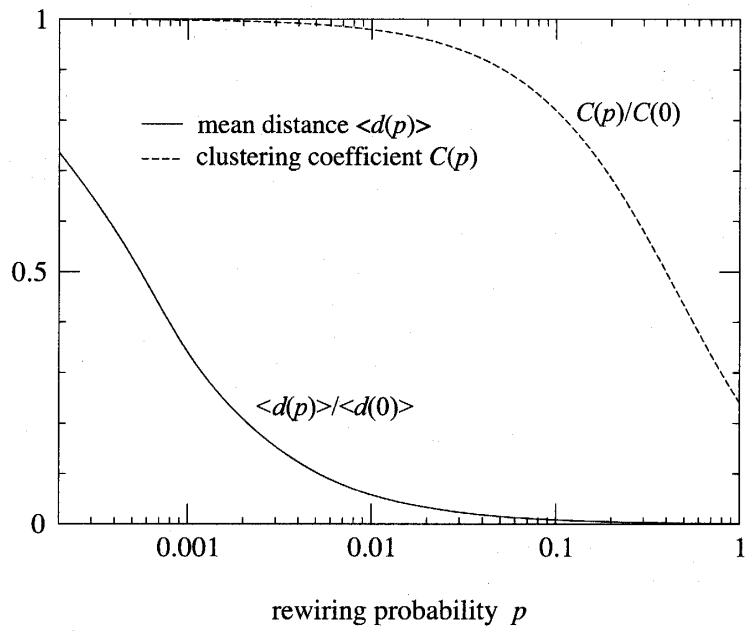

Fig. 2 From [22, Figure 2]. The relative clustering coefficient $C(p)/C(0)$ and the relative mean distances $\langle \mathrm{d}(p) \rangle / \langle \mathrm{d}(0) \rangle$ in the small-world model $WS_p(n,k)$ of Watts and Strogatz as a function of the rewiring probability p, in case of $n = 1000$ and $k = 10$. Between the extremes $p = 0$ and $p = 1$, there is a region in which clustering is high and mean vertex–vertex distance is simultaneously low.

3.2 Scale-free network

Let $p(k)$ be the fraction of vertices in the graph that have degree k, then $p(k)$ is the probability that a vertex chosen uniformly at random has degree k. Then the histogram $\{p(k)\}$ is the degree distribution for the graph. In an Erdös-Rényi random graph G_{ER}, each edge joining any chosen pair of vertices is present or absent with equal probability p and $1-p$ respectively. Hence in the random graph with N vertices and a joining probability p, the probability that a vertex v_i has $\deg(v_i) = k$ follows the binomial distribution

$$p(k) = {}_{N-1}C_k p^k (1-p)^{N-1-k},$$

for a finite case, or the Poisson distribution

$$p(k) \sim \frac{e^{-\nu}\nu^k}{k!}$$

for an inifinite case, where the average degree $\nu = (N-1)p = \langle k \rangle_{ER}$. However in many of real world networks the degree distributions follow rather

$$p(k) \sim e^{-\gamma k},$$

the power-law distributions with some constant γ, as shown in Fig.3. Networks with this power-law distribution are referred to as *scale-free networks*, now attracting much attention[2][10][18].

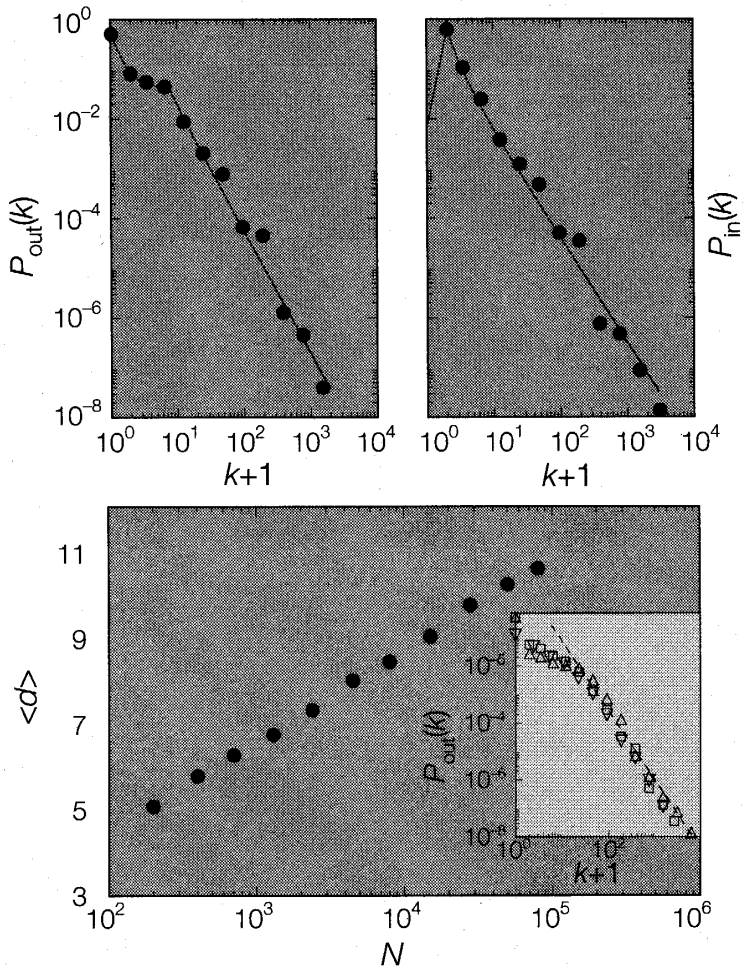

Fig. 3 From [2, Fig.1]. Distribution of links on the World-Wide Web. (a) Outgoing links (URLs found on an HTML document). (b) Incoming links (URLs pointing to a certain HTML document). Data were obtained from the complete map of the nd.edu domain. Dotted lines represent analytical fits used as input distributions in constructing the topological model of the web; the tail of the distributions follows $P(k) \approx k^{-\gamma}$, with $\gamma_{out} = 2.45$ and $\gamma_{in} = 2.1$. (c) Average of the shortest path between two documents as a function of system size, as predicted by the model. The measured $\langle d_{nd.edu} \rangle = 11.2$ agrees well with the prediction $\langle d_{3\times 10^5} \rangle = 11.6$ obtained from our model.

One of the widely accepted models for the scale-free networks was proposed by Barabási and Albert[4], who gave it as an evolution of the network. The algorithm of their construction, which we hereafter call the Barabási and Albert(BA) model, is

the preferential attachment procedure below:

1. Growth: Starting with the complete graph K_{m0} with a small number m_0. At every step add a new vertex with $m(\leq m_0)$ edges joining m different vertices already presented in the growing graph, which are chosen preferentially as follows.
2. Preferential attachment: When choosing the vertices to which the new vertex joins, we assume that the probability $\Pi(v_i)$ which a new vertex will be joined to the vertex v_i depends on the degree of v_i, say $\deg(v_i) = k_i$, such that

$$\Pi(v_i) = \frac{k_i}{\sum_{j=1}^{N} k_j}.$$

After T steps, this procedure generates a network denoted by $BA(T, m_0, m)$ with $N = m_0 + T$ vertices and mT edges. Remarkably it is proven that this simply constructed evolutive network grows up to the scale invariant network with the probability that a vertex has k edges obeying the power-law $p(k) \sim e^{-\gamma_{BA}}$ of an exponent $\gamma_{BA} = 3$ [5], independently on the parameter m_0 or m. This fact that the scale-free network has the exponent as an universal constant opens a new frontier which has been studied from the statistical physic and the random analyses[1].

The small-world network constructed by the Watts and Strogatz model does not have this scale-free property[6], so remark that the small-world property does not mean necessarily the scale-freeness.

3.3 Spectral densities

The spectral densities for the Watts and Strogatz and the Barabási and Albert models were studied numerically by Farkas *et al.*[13] and Goh *et al.*[15]. Especially the spectrum of the networks with the power-law degree distribution attracts much attentions[12][17].

The *spectral density* $\rho(\lambda)$ of a graph $G = (V, E)$ is the density of the eigenvalues $\{\lambda_i\}$ of its adjacency matrix. For a finite graph, the spectral density is defined as

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i),$$

which converges to a continuous function with the number of vertices $N \to \infty$.

If A is a real symmetric $N \times N$ matrix with averages $\langle A_{ij} \rangle = 0$ and variances $\langle A_{ij}^2 \rangle = \sigma^2$ for every $i \neq j$, and with increasing N each momenta of each $|A_{ij}|$

remains finite, then in the limit $N \to \infty$ the spectral density of $A/\sqrt{N}$ converges to the *semi-circlur distribution*[16]:

$$\rho(\lambda) = \begin{cases} (2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - \lambda^2} & \text{if } |\lambda| < 2\sigma, \\ 0 & \text{otherwise,} \end{cases}$$

which is known as *Wigner's law*. The spectral densities of the infinite Erdös-Rényi random graphs is therefore this semi-circle type.

However the semi-circle spectral density is not necessarily valid for the realistic networks. In fact, the careful numerical simulation on the BA model indicate that the bulk of the spectral density of the scale-free model has rather a triangle-like shape with top lying well above the semi-circle(Fig.4 and Fig.5).

3.3.1 Small-world network

The spectral densities $\rho(\alpha)$ of the WS model is reported to depend on the rewiring probability p [13]. For $p = 0$ the network is highly regular, so $\rho(\lambda)$ contains numerous singularities(Fig.4a). As p increases, the network is perturbed. These singularities become blurred and are transformed into high local maxima, but $\rho(\lambda)$ retains a strong skewness(Fig.4b,c) meaning that the local structure of the network is still relatively ordered, however already a very small number of rewirings can drastically change the network's global structure.

In $p = 1$ the WS model becomes very similar to the uncorrelated random graph. The only difference is that in the random graph the degree of a vertex can be any non-negative number, whereas in the WS model the minimum degree of any vertex is a positive constant $k/2$. $\rho(\lambda)$ of the WS model becomes a semicircle for $p = 1$(Fig.4d).

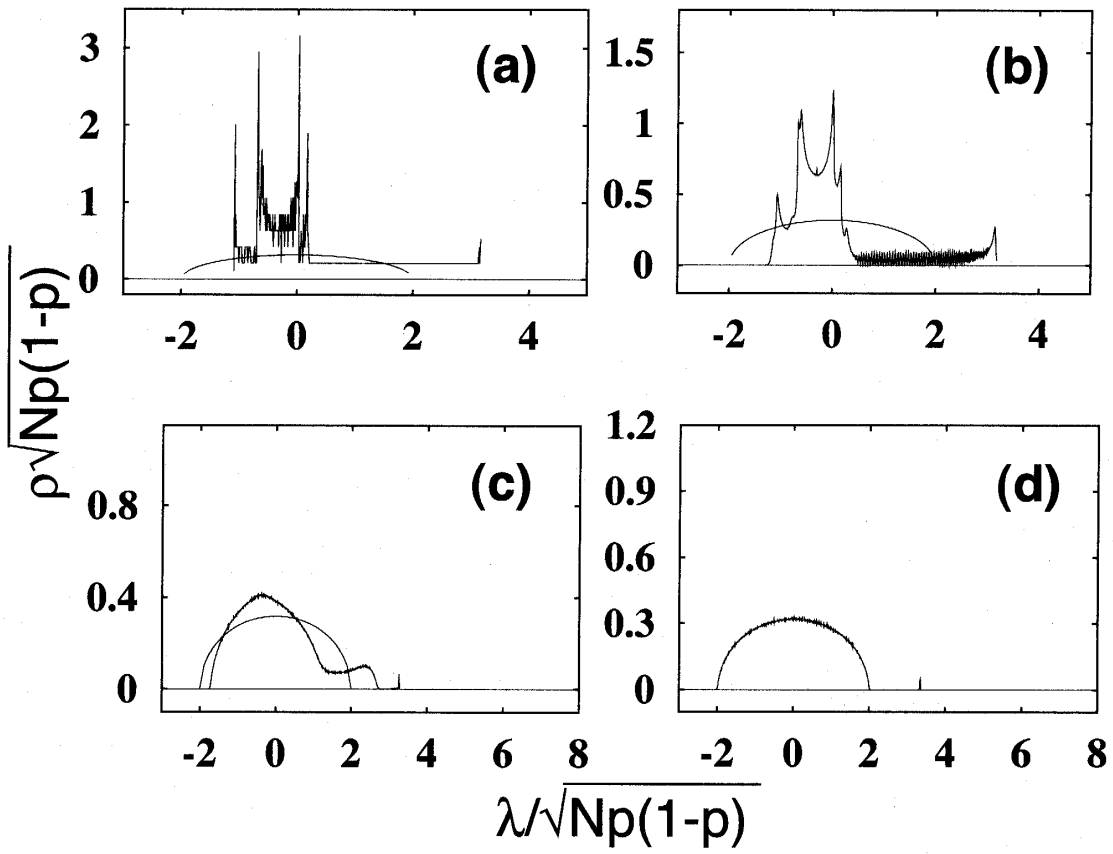

Fig. 4 After Farkas *et al.*[13]. Rescaled spectral densities of Watts and Strogatz's small-world model using the complete spectra. The solid line shows the semi-circular distribution for comparison. (a) Spectral density of the symmetric k-regular graph created from the WS model with $p = 0$, $k = 10$ and $N = 1000$. (b) For $p_{=}0.01$, the average spectral density contains sharp maxima, which are the 'blurred' remnants of the singularities of the $p_{=}0$ case. Topologically, this means, that the graph is still almost regular, but it contains a small number of impurities. In other words, after a small perturbation, the system is no more degenerate. (c) The average spectral density for the $p = 0.3$ shows that the third moment of $\rho(\lambda)$ is preserved even for very high values of p, where there is already no sign of any blurred singularity (i.e., regular structure). This means, that even though all remaining regular islands have been destroyed already, *triangles are still dominant.* (d) If $p_{\mathrm{r}} = 1$, then the spectral density converges to a semi-circle.

3.3.2 Scale-free network

Farkas *et al.* and Goh et al. found numerically that the Barabási and Albert scale-free model does not show the semi-circle spectral density[13][15]. Their numerical simulation indicate that the bulk of $\rho(\lambda)$ has a triangle-like shape with top lying well above the semi-circle and edges decaying as a power-law(Fig.5). This power-law decay is due to the eigenvectors localized on the highest degree vertecies.

As in the case of the uncorrelated random graph and unlike the SW small-world model, the largest eigenvalue λ_{max} is separated from the bulk of the spectrum and increases approximately as $N^{1/4}$. Recently Chung *et al.* obtain a theorem[12] which says that the largest eigenvalue is almost surely $\lambda_{max} = (1 + o(1))\sqrt{d_{max}}$ if $\sqrt{d_{max}} > \bar{d}\log^2 N$, where d_{max} is the maximum degree among the degrees of vertices $d_1, d_2, \ldots, d_N$ and $\bar{d}$ is the second order average degree defined by $\bar{d} = \sum_{i=1}^{N} d_i^2 / \sum_{i=1}^{N} d_i$. In a scale-free network the fraction of loops with more than four edges increases with N and their growth rate increases with the size of the loop.

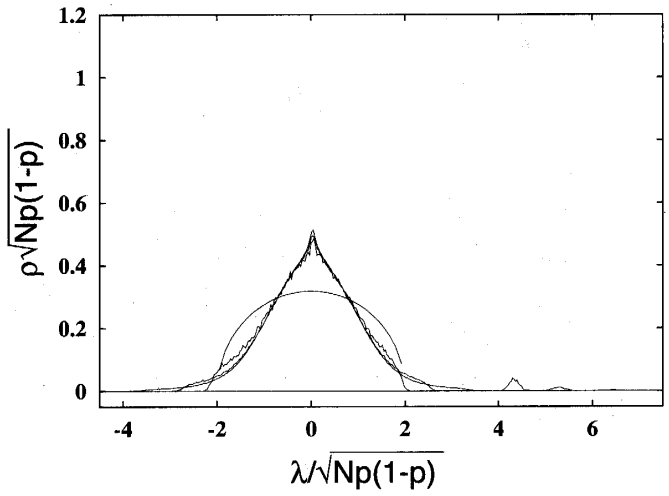

Fig. 5 After Farkas *et al.*[13]. Rescaled average spectral densities of the Barabási and Albert scale-free model with $m = m_0 = 5$, and the number of vertices $N = 100$ (—), $N = 1000$ (– –) and $N = 7000$ (- - -) vertices. The semi-circle law corresponding to random graphs is drawn for comparison. The isolated peak corresponds to the largest eigenvalue, which increases as $N^{1/4}$. Observe that (i) the central part of the scale-free graph's spectral density is triangle-like, not semi-circular and (ii) the edges show a power-law decay, whereas the the semi-circular distribution's edges decay exponentially,i.e., it decays exponentially at the edges.

4 Numerical experiments

To explore properties of the the 'small-world' and 'scale-free' network with relatively small vertices, we perform numerical experiments using the Watts and Strogatz(WS) model and the Barabási and Albert(BA) model with around 300 vertices, much smaller than the numbers used in [13][15]. All calculations are executed by *Mathematica 5.1*.

We compute the eigenvalues of the matrices by using simply the *Mathematica*'s built-in function Eigenvalues[ ] in addition to numerical value function N[ ], so their values is only approximative and furthermore more numerical errors might produce non-negligible effects for more large matrices. In general, computing numerically the correct eigenvalues of giant graphs is quite expensive and so requires a lot of the computer resources in both the memory space and the CPU time and careful algorithms to find the eigenvalues precisely.

For computation of the spectral density, we compute here the eigenvalue distribution of a graph G on the interval $I_G = [\widetilde{\xi}_{min}, \widetilde{\xi}_{max}]$ by counting number of eigenvalues $\{\widetilde{\xi}_i\}$ valued in the i-th interval $I_i (i = 1, 2 \ldots K)$ with a same width $\Delta I = \frac{1}{K}(\widetilde{\xi}_{max} - \widetilde{\xi}_{min})$, where $\widetilde{\xi}_{min}$ and $\widetilde{\xi}_{max}$ are numerically computed minimal and the maximal eigenvalues of the adjacency and/or Laplacian matrix of G respectively, $\cup_{i=1}^{K} I_i = I_G$, and the diving number K is chosen appropriately to determine the accuracy of the eigenvalue distribution on I_G for which we verified their shape by changing K. We should remember nevertheless that the numerical errors due to the floating point computing might cause some serious effects even for this approximate eigenvalue distributions.

The numerical experiments below indicate that even small-size networks show specific properties which the small-world and/or scale-free networks typically have. However specifically in case of the spectral properties for which enough size of the matrices and careful calculations of the eigenvalues should be required, we have to be aware of difficulty to say anything about for the small-size networks.

4.1 Watts and Strogatz model

4.1.1 Degree distributions

The Watts and Strogatz model $WS_p(n, k)$ with non-zero p indicate Poisson-like degree distributions even for relative small n(see Fig.6).

In a case of $p = 0$, every vertex of the WS model have same degree k. The degree distribution of the WS model drastically changes to the Poisson types onece p becomes

non-zero. The degree distributions of the WS model for any non-zero p have the same type of shapes.

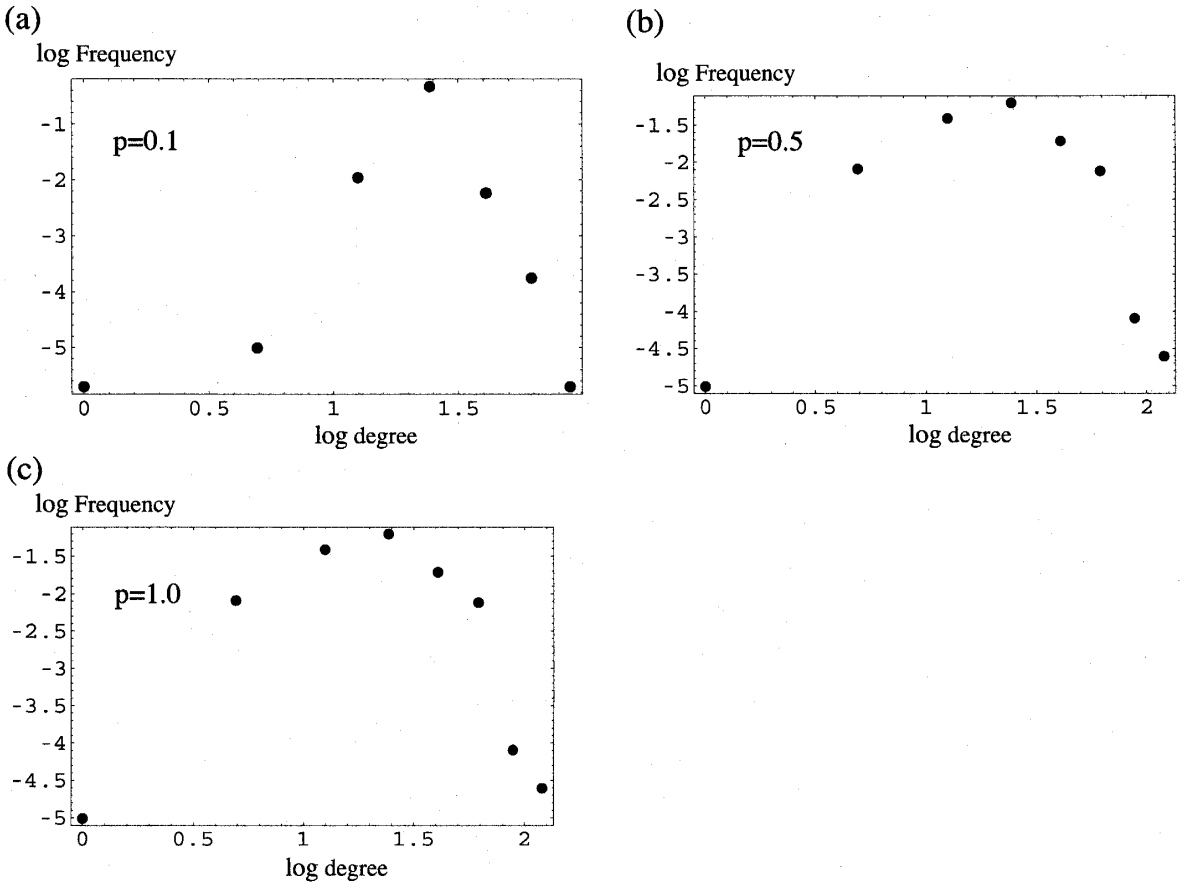

Fig. 6 The log-log-scaled degree distribution of the Watts and Strogatz model $WS_p(300, 4)$. The distributions are the Poisson type, quite different from cases of the Barabási and Albert model(*cf.* Fig.12). When $p = 0$, all vertices have same degree $k = 4$ and so the degree distribution is simple. (a) The degree distribution drastically changes to the Poisson type as long as non-zero. In this case, $p = 0.1$. The shapes of the distribution are basically same for any non-zero p. (b) $p = 0.5$. (c) $p = 1.0$.

4.1.2 The complexity

The complexity of a graph defined in Theorem 15 is one of the measures for the farness apart from tree and the amount of freedom allowing different spanning trees. The complexities of the Watts and Strogatz model $WS_p(n, k)$ grows rapidly from the

complete regular graph at $p = 0$, but saturate before the fully random rewiring graph at $p = 1$(see Fig.7).

There is a regin of p in which the complexity is high and the mean path length is simultaneously low. We remark that the non-monotonical behavior of the complexities as a function of p is newly discovered here and require further studies on the topological or algebraic structure of the WS model, because other existing one-parameter quantities such as the mean path length, diameter and number of triangles only show only the monotonicity.

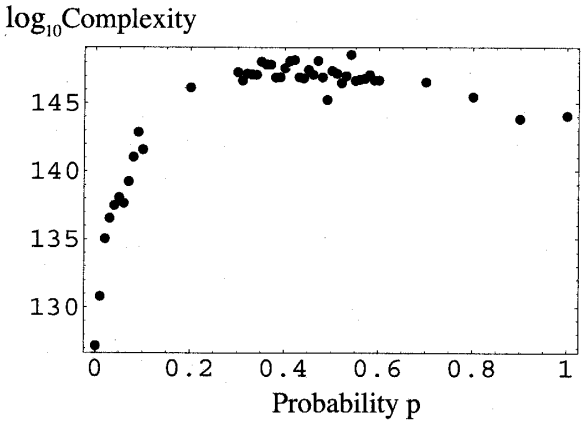

Fig. 7 The log-scale complexities of the Watts and Strogatz model $WS_p(100, 6)$ with $p = 0.0 \sim 1.0$. As the randomly rewiring probability p increases, the complexities reach maximal for intermediate p around 0.5, while they in turn decrease toward $p = 1.0$.

4.1.3 The number of triangles

The Number of triangles in the Watts and Strogatz model $WS_p(n, k)$ is gradually decreasing than the mean path length.(see Fig.8). The number of triangles is crossly related to the clustering coefficients by its definition(see Sec.2.1). Indeed as in Fig.2 and Fig.9, the the clustering coefficients have a tendency to preserves high values near $p = 0$ and so there is a region in which the clustering is high and the mean path length is simultaneously lows.

The existence of enough number of triangles for rather high values of p coincides with the results of [13] and is one of the basic property of the WS model.

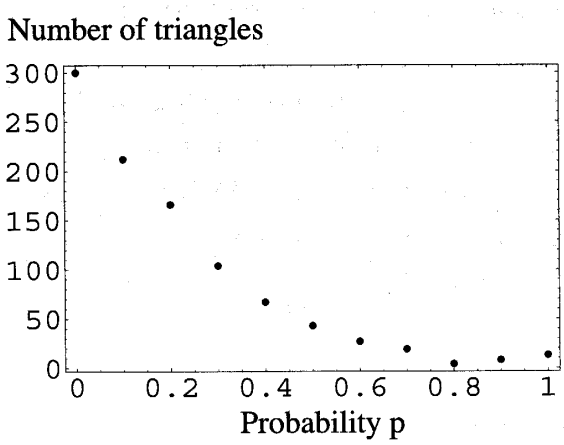

Fig. 8 Number of triangles in the Watts and Strogatz model $WS_p(300, 4)$. The numbers are decrease monotonically as p increases from 0 to 1 in a similar fashion as the cases of the relative mean path lengths in Fig.9 and Fig.2. Unlike the case of the mean path length, the number of triangles does not decrease rapidly but gradually near $p = 0$.

4.1.4 Mean path lengths and diameters

As mentioned in Sec.4.1.3, the mean path lengths decrease rapidly than the number of triangles(see Fig.9). Fig.9(a) and (b) which indicate the behavior of the diameters is similar as that of the mean path length suggests that specific topological fstructures of the network with the small-world property is not localized but scattered fully to the network by means of the random rewiring process. This result support the fact a very small number of rewirings change the network's global structure, dscribed in Sec.4.

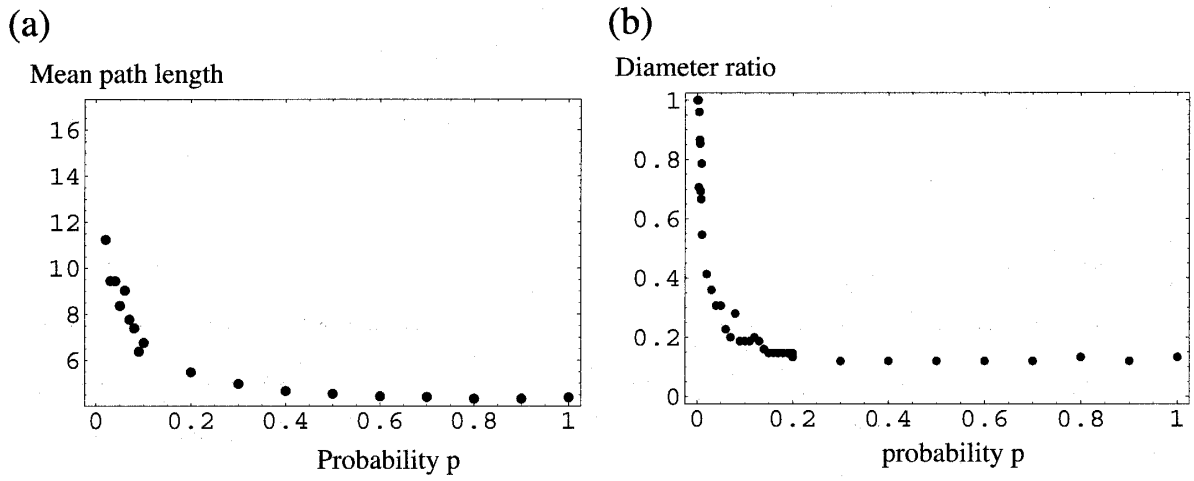

Fig. 9 Mean path lengths and diameter ratios of the Watts and Strogatz model $WS_p(1000, 10)$ from $p = 0.0 \sim 1.0$. The diameter ratio is define as a ratio of the diameter of $WS_p(n, k)$ to the diameter of $WS_0(n, k)$ which is exactly n/k. Both (a) the mean path lengths and (b) the diameter ratios are rapidly decrease near $p = 0$ contrary to the case of the number of triangles(*cf.* Fig.8).

4.1.5 Eigenvalue distribution

The eigenvalue distributions of both the adjacency and the associated Laplacian matrices are computed for the the Watts and Strogatz model $WS_p(n, k)$(see Fig.10 and Fig.11). Unfortunately the results do not show any clear-cut distinguishable distribution in p as shown in Fig.4, while in the case of the Barabási and Albert model the eigenvalue distributions are somewhat distinguishable(Fig.16). This fact might say that the eigenvalue problem of the WS model is unstable. We have to execute careful computations or prepare the enough number of vertices in the computaions.

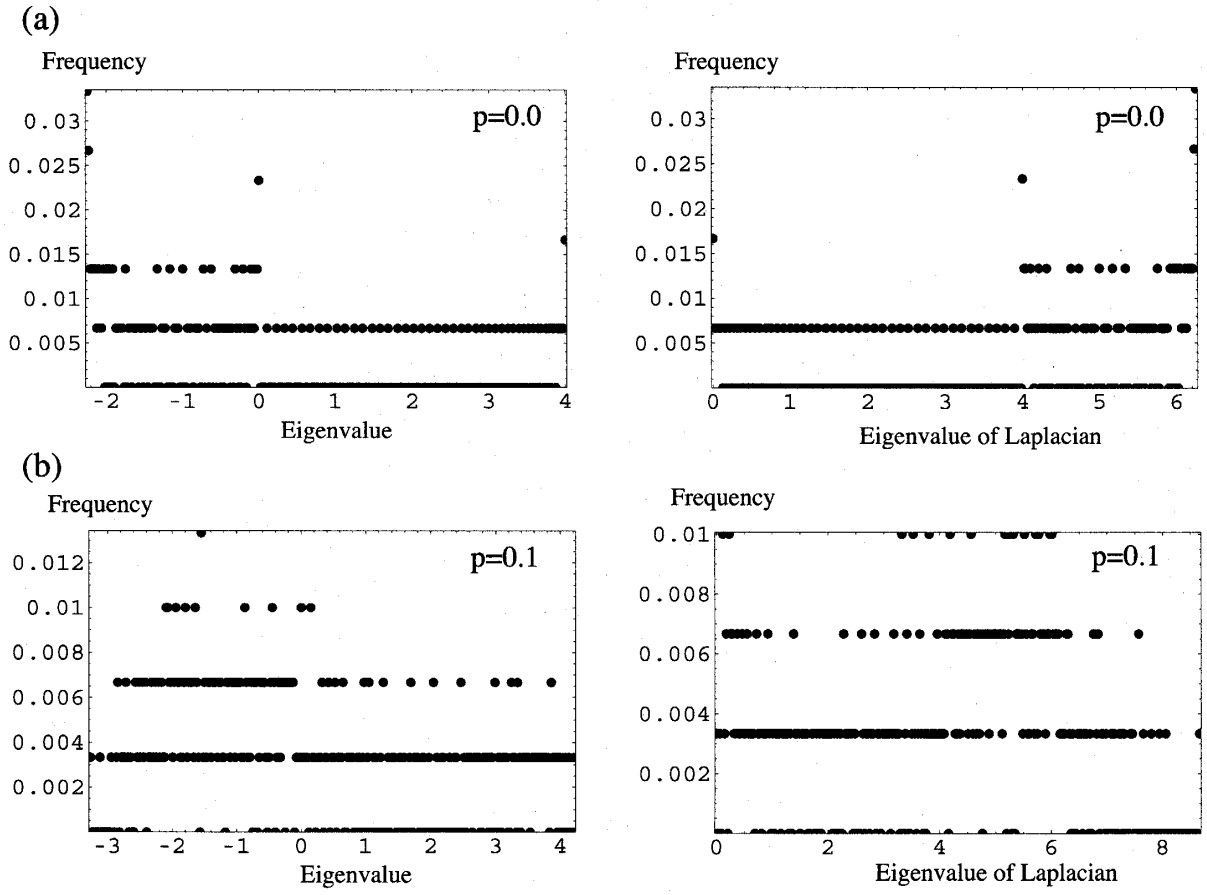

Fig. 10 (conties to Fig.11) Eigenvalue distributions for the adjacency matrices and their Laplacians of Watts and Strogatz model $WS_p(300, 4)$ for cases (a) $p = 0.0$, (b) $p = 0.1$. There are no clear-cut differences among Case (a), (b) and (c)(d) in Fig.11.

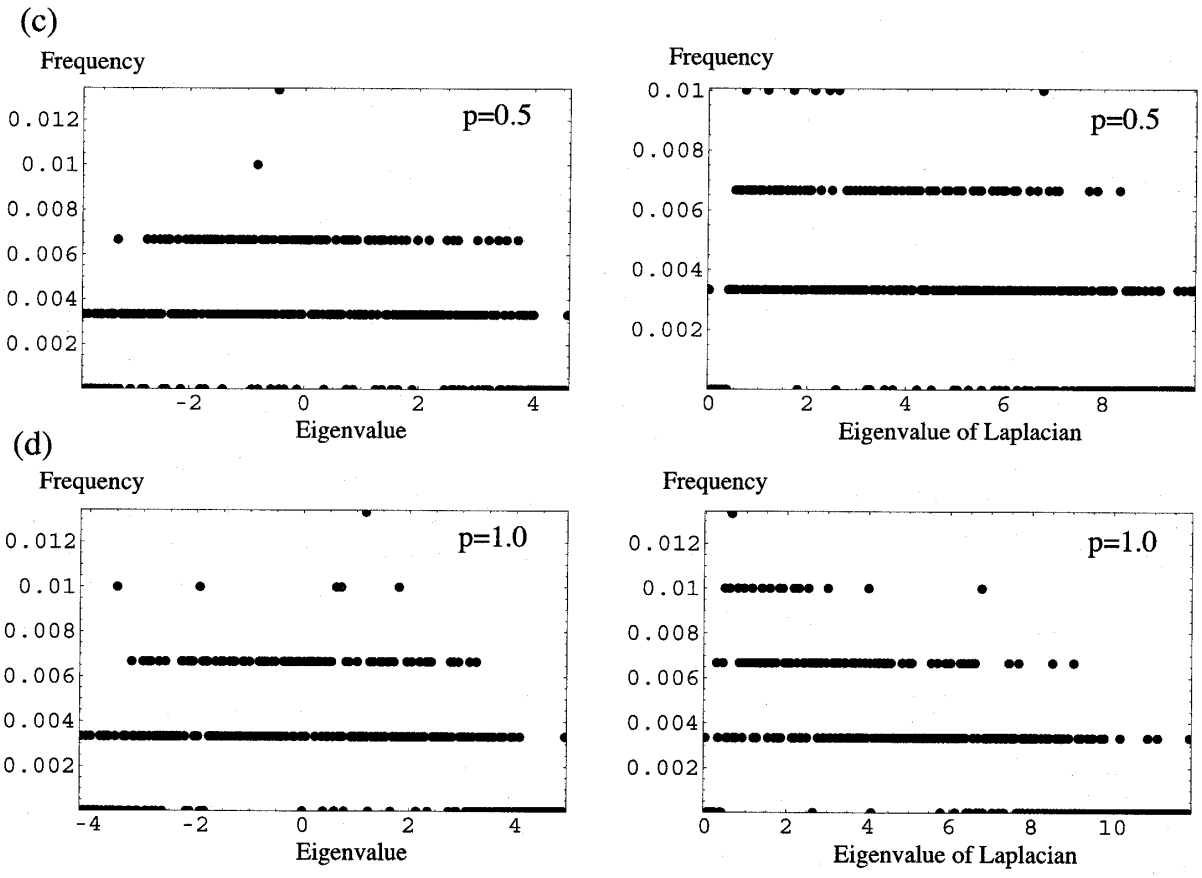

Fig. 11 (continued from Fig.10) Eigenvalue distribution for the adjacency matrices and their Laplacians of Watts and Strogatz model $WS_p(300, 4)$ for cases (c) $p = 0.5$, and (d) $p = 1.0$.

4.2 Barabási and Albert model

The Barabási and Albert model $BA(\text{step}, m_0, m)$ starts from initially the complete graph K_{m0}. At each preferential attachment process, one vertex with m edges is add to this growing-up network. If $m = 1$, the generated BA network is almost the tree, because there is no loop in the network except the initial complete graph. But for $m \geq 2$, then loops are formed and the network becomes complicated.

4.2.1 Degree distributions

The Barabási and Albert model $BA(\text{step}, m_0, m)$ have the power-law degree distribution even for small number of vertices(see Fig.12). Within our expeiments, the

exponents in the power-law distributions are same, independent on the number of attachment edges m's.

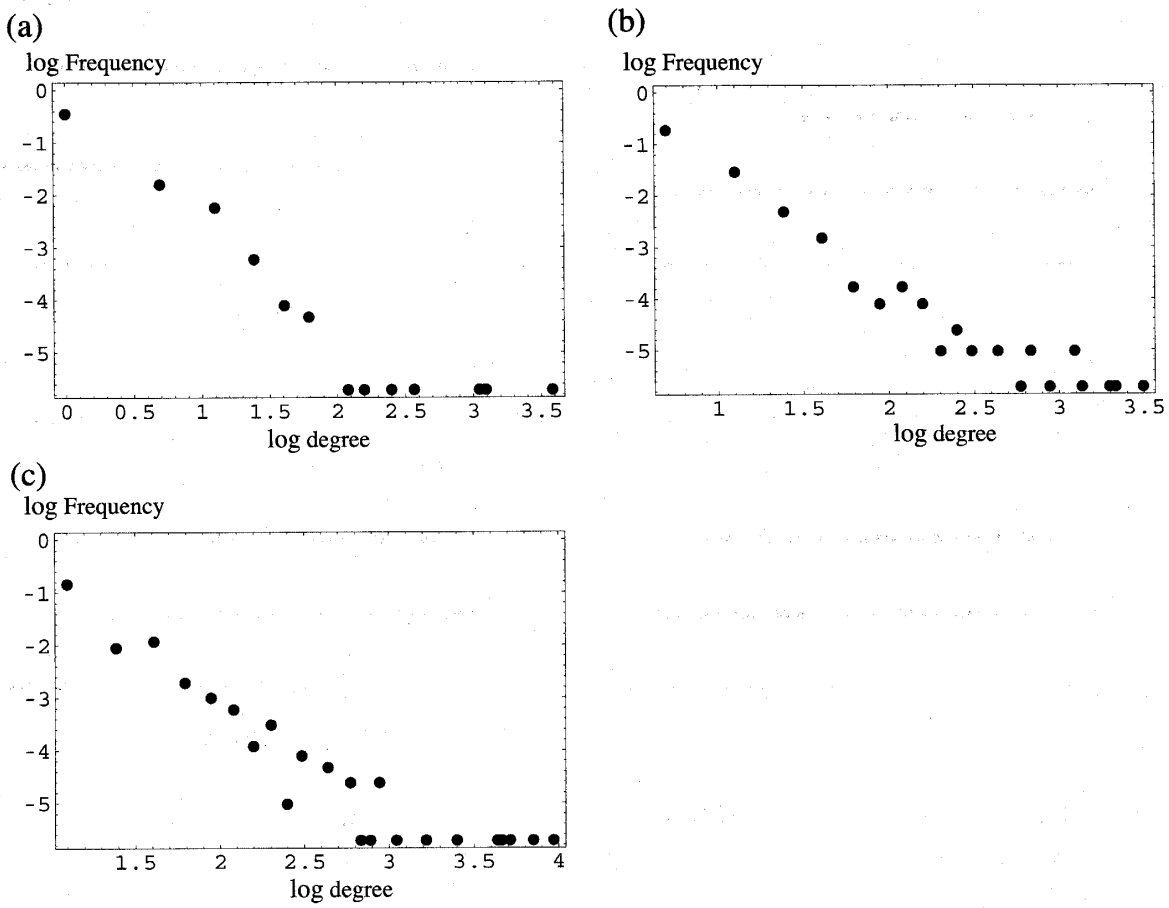

Fig. 12 Degree distributions of the Barabási and Albert model model $BA(300, 3, m)$ with (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

4.2.2 The complexity

As the preferential attachment process proceeds, the complexity of the Barabási and Albert model $BA(\text{step}, m0, m \neq 0)$ increases gradually for a while and afterward grows up dramatically (see Fig.13), in contrast to the Watts and Strogatz model $WS_p(n, k)$ where its complexity as the function of the random rewiring probability p is not monotonic(Fig.7).

If $m = 0$, the generated graph is an union of the tree and the complete graph K_{m0}. Thus the complexity is constant for every step and equal to $m_0^{m_0-2}$, the complexity of K_{m0}(*cf.* Corollary 16).

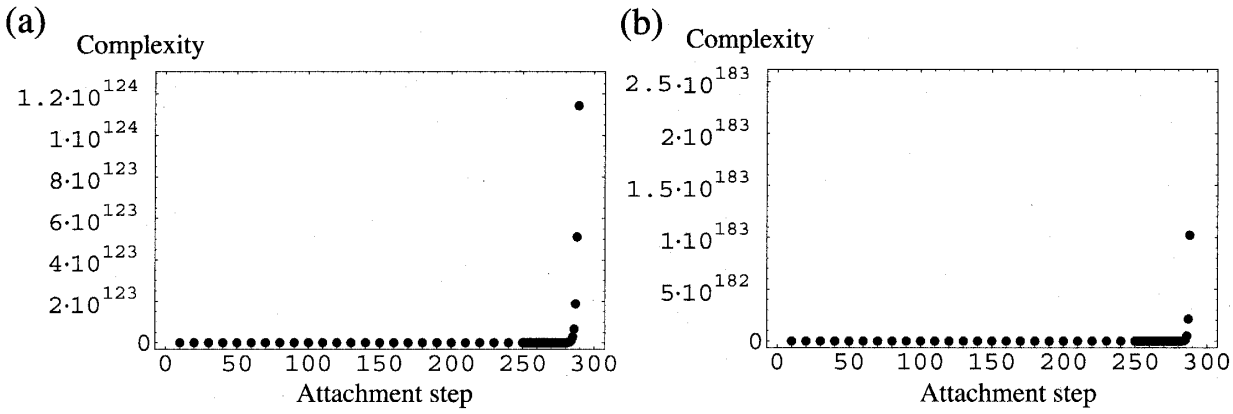

Fig. 13 Complexities of the BA model $BA(300, 3, m)$ for(a) $m = 2$ and (b) $m = 3$. As the preferential attachment process proceeds, the complexities of $BA(\text{step}, m0, m \neq 0)$ grow suddenly after some thresholds.

4.2.3 The number of triangles

The numbers of triangles of the Barabási and Albert model $BA(\text{step}, m_0, m)$ increase monotonically(see Fig.14). This monotonicity is rather normal than the case of the complexities which show the drastic growth(Fig.13).

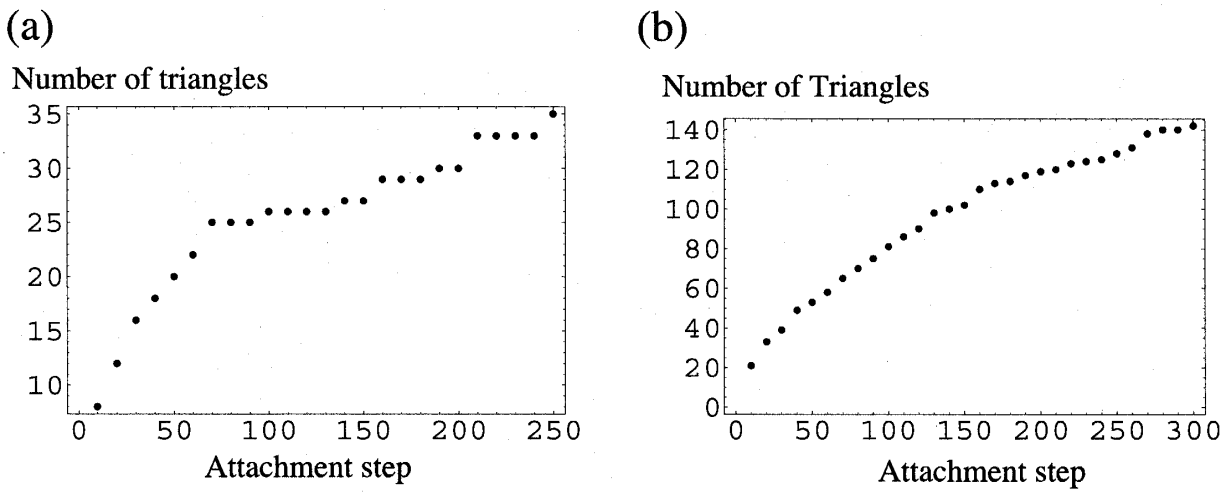

Fig. 14 Numbers of triangles in the BA model $BA(300, 3, m)$ for (a) $m = 2$ and (b) $m = 3$.

4.2.4 Mean path lengths and diameters

The mean path lengths and diameters of the Barabási and Albert model $BA(\text{step}, m_0, m)$ also increase monotonically as in the case of the number of triangles in Fig.14(see Fig.15). It is interesting that the growths of the diameters are rather stepwise than the growth of the the mean path length and that the shapes of the diagrams are seemed invariant under the appropriate scaling.

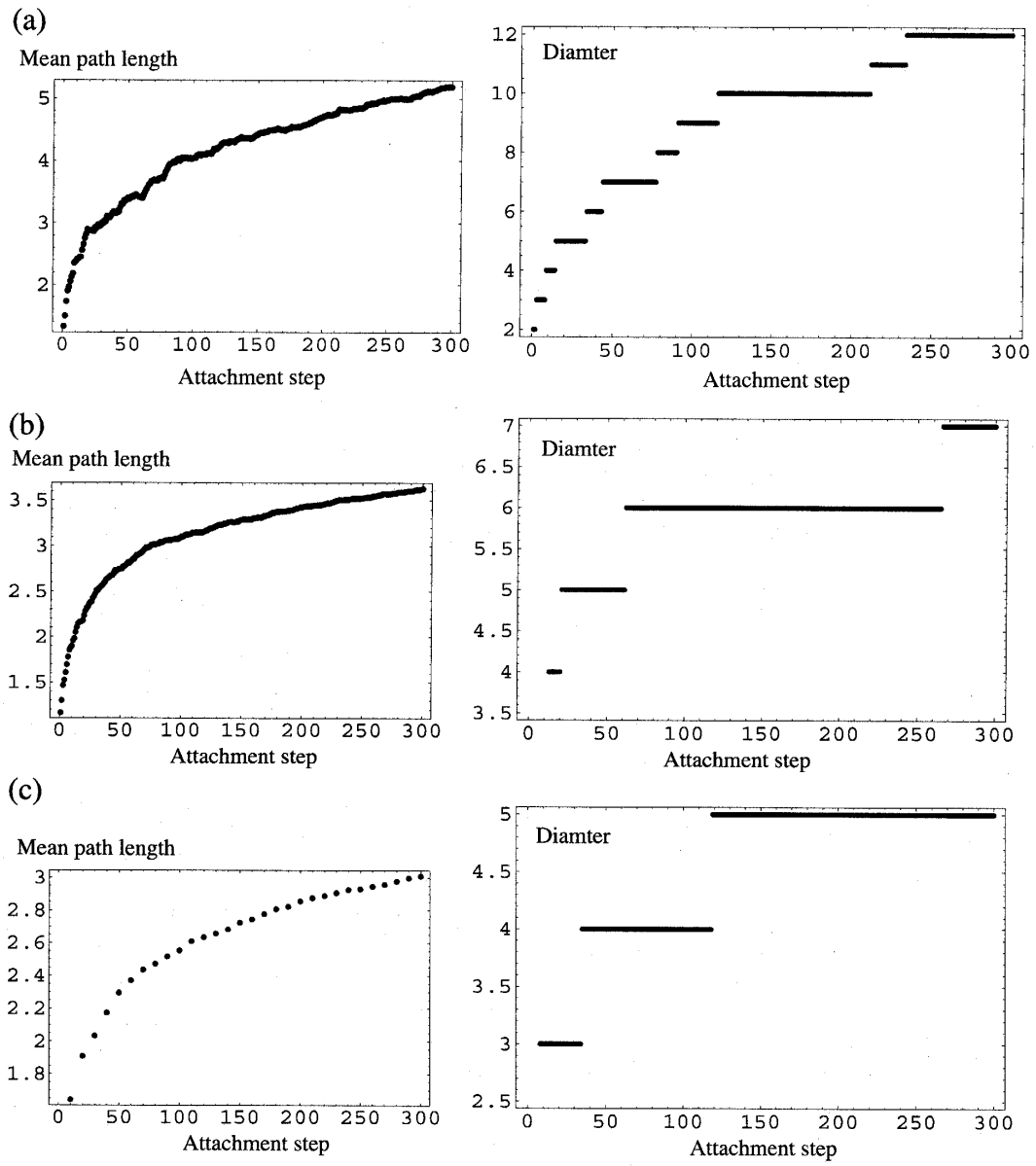

Fig. 15 Mean path lengths and diameter of the BA model $BA(300, 3, m)$ for (a) $m = 1$, (b) $m = 2$ and (c) $m = 3$.

4.2.5 Eigenvalue distribution

The eigenvalue distributions of both the adjacency and the associated Laplacian matrices are computed for the Barabási and Albert model $BA(300, 3, m)$ for for (a) $m = 1$, (b) $m = 2$ and (c) $m = 3$(see Fig.16). As in Fig.16, for even small attachment steps and larger m, in this case step $= 300$ and $m = 3$, the shape of distributions

become similar to the triangle-like as in Fig.5.

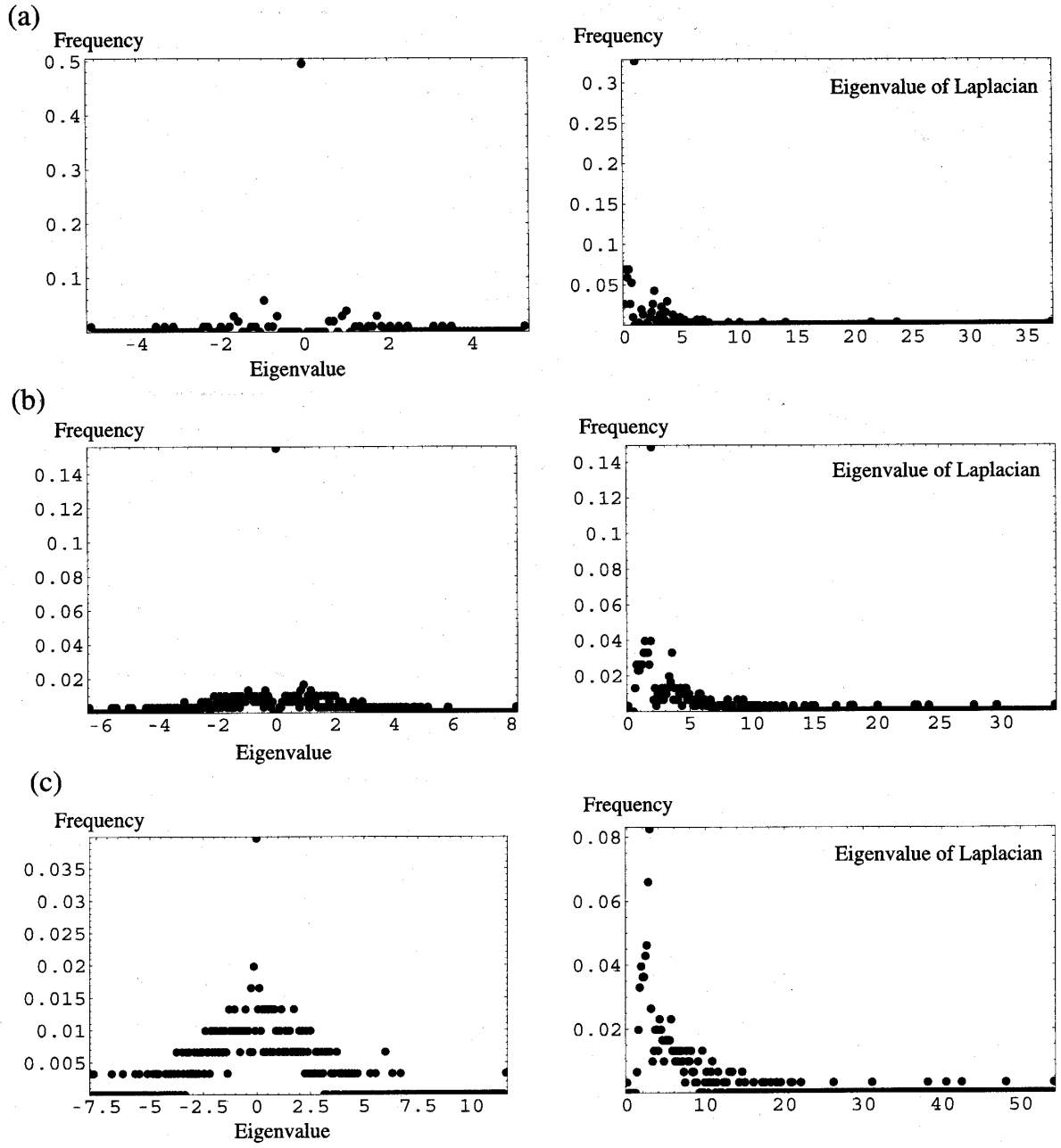

Fig. 16 Distributions Eigenvalue of the adjacency matrices and their associated Laplacian matrices of the BA model $BA(300, 3, m)$ for (a) $m = 1$, (b) $m = 2$ and (c) $m = 3$.

5 Conclusion

We have computed the degree distributions, the complexities, the numbers of triangles, the mean path lengths and the diameters, and the eigenvalue distributions for both the adjacency and the associated Laplacian matrices of the relatively small-size networks constructed by the Watts and Strogatz(WS) and the Barabási and Albert(BA) models.

We could verify that many of the small-world and scale-free properties are hold even for these small networks. Remark that these computed properties are not really relevant to the eigenvalues but directly derived from the matrices of the networks without any numerical errors.

However as far as the eigenvalue distributions of the WS model, unfortunately there are no obvious differences among the distributions with different values of the random rewiring probability p, while the careful and large-size simulations show that there is apparent differences among the distributions as in Fig.4. On the other hand, the eigenvalue distribution of the BA model become similar to the shape obtained by the careful and large-size simulations as in Fig.5.

Thus the fact above requires to study the differences of the algebraic structures between these network models exhaustively. In fact, the WS model has specific regions of p in which two opposite properties are simultaneously coexistent; the region with high clustering and simultaneously low mean path length, and the region with low mean path length and simultaneously high complexity, etc. The complexities of the WS model behaves not monotonically and take maximal value around $p = 0.5$. This behavior is newly discovered here.

References

[1] R. Albert and A.-L. Barabási, *Statistical mechanics of complex networks*, Review of Modern Physics, 74(2002), pp.47-97.

[2] R. Albert, H. Jeong and A.-L. Barabási, *Diameter of the world-wide web*, Nature 401 (1999), 130.

[3] A.-L. Barabási, *Linked: The New Science of Networks*, Perseus Press, New York, 2002.

[4] A.-L. Barabási and R. Albert, *Emergence of scaling in random networks*, Science 286 (1999), pp.509-512.

[5] A.-L. Barabási, R. Albert and H. Jeong, *Mean-field theory for scale-free random networks*, Physica A (1999), pp.173?187.

[6] A. Barrat and M. Weigt, *On the properties of small-world network models*, Europ. Phys. J. B 13 (2000), 547.

[7] L.W. Beineke and R.J. Wilson (eds.), *Topics in Algebraic Graph Theory*, Cambridge Univ. Press, 2004.

[8] No. Biggs, *Algebraic Graph Theory–Second Edition*, CAmbridge Univ.Press, 1993.

[9] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1991.

[10] S.N. Dorogovtsev and J.F.F. Mendes, *Evolution of Networks*, Oxford Univ. Press, 2003.

[11] S. Bornholdt and H.G. Schuster(eds), *Handbook of Graph and Networks–From the Genome to the Internet*, WILEY-VCH, 2003.

[12] F. Chung, L. Lu and V. Vu, *Eigenvalues of Random Power law Graphs*, Annals of Combinatorics 7 No.1 (2003), pp.21 - 33.

[13] I.J. Farkas, I. Derényi, A.-L. Barabási and T. Vicsek, *Spectra of "Real-World" Graphs: Beyond the Semi-Circle Law*, Phisical Review E64 (2001), pp.1-12.

[14] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, 2001.

[15] K.-I. Goh, B. Kahng, and D. Kim, *Spectra and eigenvectors of scale-free networks*, Phys. Rev. E 64, 051903 (2001)

[16] M.L. Mehta, *Random Matrices*, Academic Press, New York, 1991.

[17] M. Mihail and C.Papadimitriou, *On the Eigenvalue Power Law*, in '*Randomization and Approximation Techniques : 6th International Workshop, RANDOM 2002*', Lecture Notes in Computer Science 2483, Springer-Verlag, 2002.

[18] M. E. J. Newman, *The structure and function of complex networks*, SIAM Review 45 (2003), 167-256.

[19] D. Watts, *Small Worlds: The Dynamics of Networks Between Order and Randomness*, Princeton Univ Press, 2004.

[20] D.J. Watts, *Six Degrees – The sience of a connected age*, W.W. Norton & Company, New York, 2003.

[21] S. Wasserman and K. Faust, *Social Network Analysis–Method and Applications*, Cambridge Univ.Press, 1994.

[22] D.J. Watts and S.H. Strogatz, *Collective dynamics of small-world network*, Nature 393 (1998), 440.

[23] Yoshiko Takenaka, *Linear Algebraic Graph Theory* (in Japanese), Baifuhkan, Tokyo, 1989.