【Research Note】

Dagum's Gini Coefficient Decompositions and its Extension by Deutsch and Silber :

A Illustrative Two Period Numerical Example.

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ABSTRACT

This research note presents a review of the methodology developed by Deutsch and Silber (1999), which extended the decomposition of the Gini index as described by Dagum (1997). Using a simple two-income group data, this paper focuses on three things; reviewing the contributions of different income group to Gini index, presenting the discrete version of the development of Dagum's (1997) decomposition, and developing the changes in the components of the decomposition of the Gini index.

For this explanation, a sizable numerical example is used, which hopefully facilitate the work of other researchers when applying this methodology for the analysis of various real world income data sets.

KEYWORDS

Gini coefficient, Decomposition of Gini coefficient, Numerical Example.

CONTENTS

Dagum による Gini 指標の分解と Deutsch and Silber による拡張：2 期間分析の数値例

岡 村　與 子

要　旨

この研究ノートは Deutsch and Silber (1999)による Dagum(1997)の Gini 指標の分解の発展形の展開方法を詳しく考察している。単純化のために 2 グループからなる所得のデータの数値例を用いながら、このノートでは３つのことを行っている。それらは、グループごとの Gini 指標への寄与分析を概観すること、Dagum による分解方法の計算を離散的な観測データを想定して詳細に示すこと、そして Gini 指標が変化する場合に、それぞれの分解項目がどのように変化に対して寄与するかを分析することである。これらの分析方法を示すために、把握しやすいように数値例を用いているが、このノートによりこれらの分析方法がより多くの現実に観測される所得などの不平等分析に活用されるようことを願っている。

キーワード　ジニ指標（係数），　ジニ指標の分解，　数値例

JEL classification code: D31, D63

Introduction[1]

As the inequality among individuals in many societies is perceived to be increasing, attention is being paid by social scientists in different fields to quantify the degree of this inequality. In exploring the causes or consequences of such inequalities, not only the measurement of inequality among individuals in a population (called vertical inequality), but also the measurement of inequality among sub-groups of such populations (called

[1] The author would like to express gratitude to Professor Silber for providing notes (Silber, 2014) to explain the contribution of each group to the decompositional components. The author is responsible for all mistakes that may exist.

horizontal inequality) is becoming more important [2].

In addition, while there are many methods for evaluating inequality; the Gini index is still one of the most commonly used coefficient for measuring such inequalities. The Gini index is defined as follows. When there are an individual income data, denoted as Y_i and the mean of the entire data as Y_m, the Gini index is computed by;

$$I_G = \frac{\Delta}{2n^2 Y_m} \qquad where \quad \Delta = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|Y_i - Y_j\right| \qquad \text{(Eq. 1)}$$

Although (Eq. 1) has been computed for many income data, an elaboration of the Gini index for evaluating horizontal inequality is also desirable. Particularly, those who seek explanations of the increase or decrease of inequality in the subject societies or economies would like to examine the sub-groups of the original populations and their relationships to the entire populations. This paper addresses the needs and interests of those who are concerned about such inequalities using the decomposition of the Gini index. The decomposition of the Gini index consists of three parts as shown in the (Eq. 2).

$$I_G = I_w + I_B + I_V \qquad \text{(Eq. 2)}$$

In this paper, even though many authors have developed different computational formulae for (Eq. 1), one developed by Dagum (1997), later extended by Deutsch and Silber (1999) is carefully examined. Usually, a decomposition of the Gini Index (I_G) is consists of the within group inequality (I_W), the between group inequality (I_B) and the transvariational inequality (I_V) as shown in (Eq.2), however; the decomposition of the index examined in this research note is based on a simple artificial numerical example.

In demonstrating the computational procedure by following Deutsch and Silber (1999), for the sake of simplicity, it is assumed that there are only two overlapping income subgroups, a 'high income group' (denoted by H) and a 'low income group' (denoted by L), where group H exhibit a higher income average than L. To those several computational methods associated with the decomposition are applied. In the first part, the direct computation of (Eq. 2), the contribution of each sub-group to components of (Eq. 2), and the Dagum's presentation of the decomposition are shown. Then, in the second part, another set of artificial data is presented, again consisting of two subgroups, showing how the change in the sub-components of the decomposition contributes to the change in the decompositional parts of the Gini ratio.

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2 The notion of horizontal inequality was proposed by Stewart (2002). Stewart emphasized the importance of group well-being for individual welfare, and consequently for social stability.

Part I:　Two Sub-groups Example

In this Part, the basic two sub-groups (high and low income group) are presented and then the Gini ratio for the entire data and the decomposed Gini are computed.　In addition, the contribution of each income group to each components of the decomposition is presented and computed for the numerical example. Finally, the discrete version of Dagum's (1997) presentation of decomposition is also presented and demonstrated.

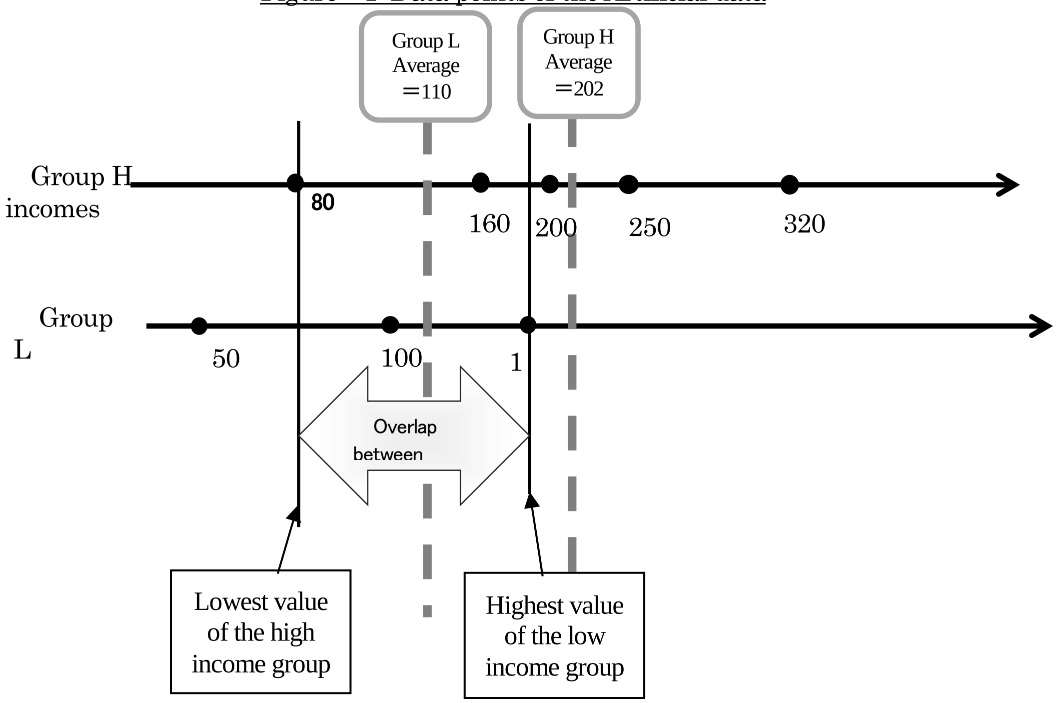
I – 1　Presentation of One-period Example

In order to easily see how the decomposition is computed, a data set was created as shown below.　The artificial income data consists of eight data points. Each of the data points is categorized into either the high income group (H) or the low income group (L). The values of the data points with group division are shown as;

Group H={80,160,200,250,320}

Group L={50,100,180}

Figure　1: Data points of the Artificial data



The average income for the group H is $Y_{Hm} = 202$, and for group $Y_{Lm} = 110$. Also, the average income of the entire data is Y_m=167.5.　Also, the "location" of the data can be illustrated along the numerical line for each group as shown in Figure 1.

To compute the Gini index for the entire population, the differences between all possible pair of the data should be calculated. The differences needed for the computation are organized as shown in Table 1. In this table, the data points are presented in the italic-bold face both in the second row and the second column. The values of the data are ordered first according to the group, and then according to the ascending order within the group. The difference values are computed by subtracting the data in the row from in the column.

Table 1: Income Differences between All Pair of Observations*.

		Group H Data					Group L Data		
		80	***160***	***200***	***250***	***320***	***50***	***100***	***180***
Group H Data	***80***	0	-80	-120	-170	-240	30	-20	-100
	160	80	0	-40	-90	-160	110	60	-20
	200	120	40	0	-50	-120	150	100	20
	250	170	90	50	0	-70	200	150	70
	320	240	160	120	70	0	270	220	140
Group L Data	***50***	-30	-110	-150	-200	-270	0	-50	-130
	100	20	-60	-100	-150	-220	50	0	-80
	180	100	20	-20	-70	-140	130	80	0

*The artificial income data are shown in italic face font. Differences are computed by subtracting the data in the column from data in the row.

In order to compute the Gini index for the all income data, the absolute values of the all elements in Table 1 are summed. This gives,

$$\Delta = \sum_{i=1}^{8} \sum_{j=1}^{8} |Y_i - Y_j| = 6120 \qquad \text{(Eq. 3)}$$

Where Y_i (or Y_j) denote the income of ith (or jth) member of the entire population[3]. The Gini index of the example data is computed as;

$$I_G = \frac{\Delta}{2n^2 Y_m} = \frac{1}{2}\frac{1}{(8)^2}\frac{1}{(167.5)}(6120) = (10^{-5} \times 4.6641) \times (6120) = 0.2854 \qquad \text{(Eq. 4)}$$

Further, in evaluating the decomposition of Gini index as (Eq. 2), Δ can be seen in two parts; the first part computed from the differences within the same groups, another from the differences across the groups. So, the sum of the within group differences is described

[3] For the entire data, the index i or j is used. So, i or j runs from 1 to n. For this artificial data, n=8, and $Y_1 = 80, Y_2 = 160, \cdots Y_8 = 180$.

as $\Delta_W =$ {absolute values of the differences within H group data}+{ absolute values of the differences within L group data}. Using *S* for the index of high income group H, and *R* for the index of low income group L,

$$\Delta_W = \sum_{h=H,L} \left\{ \sum_{S=1}^{n_h} \sum_{S'=1}^{n_h} |Y_{hS} - Y_{hS'}| \right\}$$
$$= \sum_{S=1}^{n_H} \sum_{S'=1}^{n_H} |Y_{HS} - Y_{HS'}| + \sum_{R=1}^{n_L} \sum_{R'=1}^{n_L} |Y_{LR} - Y_{LR'}| \qquad \text{(Eq. 5)}$$

Then for data in the example, the value is;

$$\Delta_W = \sum_{S=1}^{5} \sum_{S'=1}^{5} |Y_{HS} - Y_{HS'}| + \sum_{R=1}^{3} \sum_{R'=1}^{3} |Y_{LR} - Y_{LR'}| \qquad \text{(Eq. 6)}$$
$$= 2280 + 520 = 2800$$

The difference between Δ and Δ_W is then specified as a summations of the across differences.

$$\Delta_A = \Delta - \Delta_W = 2\left\{ \sum_{S=1}^{n_H} \sum_{R=1}^{n_L} |Y_{HS} - Y_{LR}| \right\} \qquad \text{(Eq. 7)}$$

The differences in (Eq. 7) are computed by subtracting the low mean group elements from the high mean group elements only. To maintain the identity, $\Delta = \Delta_W + \Delta_A$, the sum of the absolute values of subtracted values are multiplied by 2. For the numerical example above, $\Delta_A = 2 \times 1660 = 3320 = 6120 - 2800$

To expand the absolute values in (Eq. 7), the elements in Table 1 are sorted according to the sign. When $Y_{HS} > Y_{LR}$ then, $|Y_{HS} - Y_{LR}| = Y_{HS} - Y_{LR}$. On the other hand, when $Y_{HS} < Y_{LR}$,$|Y_{HS} - Y_{LR}| = Y_{LR} - Y_{HS}$. According to the sign, values of the difference in (Eq. 7) are separated.

For $Y_{HS} > Y_{LR}$
$$\widetilde{\Delta_d} = \sum_S \sum_R (Y_{HS} - Y_{LR}) \qquad \text{(Eq. 8)}$$

For $Y_{HS} < Y_{LR}$
$$\widetilde{\Delta_p} = \sum_S \sum_R (Y_{LR} - Y_{HS}) \qquad \text{(Eq. 9)}$$

Let $\Delta_d = 2\widetilde{\Delta_d}$ and $\Delta_P = 2\widetilde{\Delta_P}$, then $\Delta_A = \Delta_d + \Delta_p$ is hold.

The sub-table extracted from Table 1 for the differences between Group H data and

Group L data can be shown as;

Table 2: Difference between Group H data and Group L data†:

		Group L Data		
		50	**100**	**180**
Group H Data	**80**	30	-20	-100
	160	110	60	-20
	200	150	100	20
	250	200	150	70
	320	270	220	140

† This table is north-west sub-table from Table 1.
Total of the absolute value of the elements surrounded by the bold lines is 1660.

Therefore, the positive elements in Table 2 are summed up to give,

$$\widetilde{\Delta_d} = \sum_S \sum_R (Y_{HS} - Y_{LR}) = \qquad \text{(Eq. 10)}$$

$$30 + 110 + 60 + 150 + 100 + 20 + 200 + 150 + 70 + 270 + 220 + 140 = 1520$$

and

$$\widetilde{\Delta_p} = \sum_R \sum_S (Y_{LR} - Y_{HS}) = 20 + 100 + 20 = 140 \qquad \text{(Eq. 11)}$$

So, $\Delta_A = \Delta_d + \Delta_p = 2(1520 + 140) = 3320$. Adding to $\Delta_W = 2800$, Δ =6120 is confirmed. Furthermore, it can be written as $\Delta_A = (\Delta_d - \Delta_p) + 2\Delta_p$. Substituting this expression into $\Delta = \Delta_W + \Delta_A$ gives:

$$\Delta = \Delta_W + (\Delta_d - \Delta_p) + 2\Delta_p \qquad \text{(Eq. 12)}$$

Substituting (Eq. 12) into I_G in (Eq. 1), gives:

$$I_G = \frac{\Delta}{2n^2Y_m} = \frac{1}{2n^2Y_m}\{\Delta_W + (\Delta_d - \Delta_p) + 2\Delta_p\} \qquad \text{(Eq. 13)}$$

Let

$$I_W = \frac{\Delta_W}{2n^2Y_m}$$

$$I_B = \frac{(\Delta_d - \Delta_p)}{2n^2Y_m} \qquad \text{(Eq. 14)}$$

$$I_V = \frac{2\Delta_p}{2n^2Y_m}$$

Then $I_G = I_W + I_B + I_V$ is hold. For this numerical example,

$$I_W = \frac{2800}{2(64)(167.5)} = 0.1306$$

$$I_B = \frac{2(1520-140)}{2(64)(167.5)} = \frac{2760}{2(64)(167.5)} = 0.1287$$

$$I_V = \frac{2\Delta_p}{2n^2Y_m} = \frac{560}{2(64)(167.5)} = 0.0261$$

Therefore, the summation of these terms results in $I_G = \frac{6120}{2(64)(167.5)} = 0.2854$. This result is thus also confirmed.

I – 2　The Contribution of Each Subgroup

Following the methodology of Deutsch and Silber (1999), contribution of each subgroup H and L is specified and computed. From equation (Eq. 5) and the first equation in (Eq. 14), the contribution of the within difference terms are defined as;

$$I_W = \frac{\Delta_W}{2n^2Y_m} = \frac{\{\ \sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}|Y_{HS} - Y_{HS'}| + \sum_{R=1}^{n_L}\sum_{R'=1}^{n_L}|Y_{LR} - Y_{LR}|\}}{2n^2Y_m} \qquad \text{(Eq. 15)}$$

Therefore, by letting

$$C_{WH} = \frac{1}{2n^2Y_m}\sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}|Y_{HS} - Y_{HS'}|$$
$$C_{WL} = \frac{1}{2n^2Y_m}\sum_{R=1}^{n_L}\sum_{R=1}^{n_L}|Y_{LR} - Y_{LR'}| \qquad \text{(Eq. 16)}$$

In the example, $C_{WH} = \frac{2280}{2(8)^2(167.5)} = 0.1063$ and $C_{WL} = \frac{520}{2(8)^2(167.5)} = 0.0243$.

$$I_W = C_{WH} + C_{WL} = 0.1063433 + 0.024254 = 0.1306 \qquad \text{(Eq. 17)}$$

Further, from equation (Eq. 8), (Eq. 9) and (Eq. 14), Δ_d and Δ_p are defined holding the ordering by the mean income levels. The contribution of the each group to $\widetilde{\Delta_d}$ and $\widetilde{\Delta_p}$ is specified as the following[4].

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[4] When there are three groups with high, middle and low mean incomes, the equation (Eq. 8)

The contribution of the high income group to the $\widetilde{\Delta_d}$ is:

$$C_{\widetilde{\Delta d}H} = \sum_S \sum_R (Y_{HS} - Y_{LR}) \quad for \quad Y_{HS} > Y_{LR} .$$

The contribution of the low income group to the $\widetilde{\Delta_d}$ is zero: $C_{\widetilde{\Delta d}L}=0$

The contribution of the high income group to the $\widetilde{\Delta_p}$ is zero. $C_{\widetilde{\Delta p}H}=0$

And, the contribution of the low income group to the $\widetilde{\Delta_p}$ is:

$$C_{\widetilde{\Delta p}L} = \sum_R \sum_S (Y_{LR} - Y_{HS}) \quad for \quad Y_{LR} > Y_{HS} .$$

Therefore, contributions of high income group to the between group inequality can be presented as specified by;

for Δ_d

$$C_{\Delta d,H} = \frac{2C_{\widetilde{\Delta d}H}}{2n^2 Y_m} = \frac{2}{2n^2 Y_m}\left\{\sum\nolimits_S \sum\nolimits_R (Y_{HS} - Y_{LR}) \quad for \quad Y_{HS} > Y_{LR}\right\} \qquad \text{(Eq. 18)}$$

and

$$C_{\Delta d,L} = 0$$

Similarly,

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becomes

$$\widetilde{\Delta_d} = \left\{\sum\nolimits_S \sum\nolimits_Q (Y_{HS} - Y_{MQ}) \ for \ Y_{HS} > Y_{MQ}\right\} +$$

$$\left\{\sum\nolimits_S \sum\nolimits_R (Y_{HS} - Y_{LR}) \ for \ Y_{HS} > Y_{LR}\right\} +$$

$$\left\{\sum\nolimits_Q \sum\nolimits_R (Y_{MQ} - Y_{LR}) \ for \ Y_{MQ} > Y_{LR}\right\}$$

Out of these terms, the contribution of the highest mean income group is specified as the first two terms where the values of the highest income contributes positively to the $\widetilde{\Delta_d}$. Similarly, the last term is specified as the contribution of the middle income group to $\widetilde{\Delta_d}$ and the contribution of the lowest income is zero. Likewise, for (Eq. 9),

$$\widetilde{\Delta_p} = \left\{\sum\nolimits_Q \sum\nolimits_S (Y_{MQ} - Y_{HS}) \ for \ Y_{MQ} > Y_{HS}\right\} +$$

$$\left\{\sum\nolimits_R \sum\nolimits_S (Y_{LR} - Y_{HS}) \ for \ Y_{LR} > Y_{HS}\right\} +$$

$$\left\{\sum\nolimits_R \sum\nolimits_Q (Y_{LR} - Y_{MQ}) \ for \ Y_{LR} > Y_{MQ}\right\}$$

the contribution of the highest income group is zero while the contribution of middle income group to $\widetilde{\Delta_p}$ is specified in the first term and the contribution of the lowest income group is the last two terms in $\widetilde{\Delta_p}$.

for Δ_P

$$C_{\Delta p,H} = 0$$

and (Eq. 19)

$$C_{\Delta p,L} = \frac{2C_{\widetilde{\Delta p},L}}{2n^2 Y_m}\left\{\sum_R \sum_S (Y_{LR} - Y_{HS}) \quad for \quad Y_{LR} > Y_{HS}\right\}$$

The contribution of the each group to the between inequality is

$$I_B = \frac{2(\widetilde{\Delta_d} - \widetilde{\Delta_p})}{2n^2 Y_m}$$

$$= \frac{2}{2n^2 Y_m}\left[\left\{\sum_S \sum_R (Y_{HS} - Y_{LR}) \quad for \quad Y_{HS} > Y_{Lr}\right\} - \sum_R \sum_S (Y_{LR} - Y_{HS}) \quad for \quad Y_{HS} < Y_{Lr}\right]$$ (Eq. 20)

$$= \{C_{\Delta d,H} + C_{\Delta d,L}\} - \{C_{\Delta p,H} + C_{\Delta p,L}\}$$

$$= \{C_{\Delta d,H} - C_{\Delta p,H}\} + \{C_{\Delta d,L} - C_{\Delta p,L}\} = C_{BH} + C_{BL}$$

For the case of the two subgroup example above, the contribution of each group to the between group inequality

$$C_{BH} = \frac{2(1520-0)}{2(8)^2(167.5)} = 0.1418 \ \text{ and } \ C_{BL} = \frac{2(0-140)}{2(8)^2(167.5)} = -0.0131$$

Therefor the result is verified as; $I_B = C_{BH} + C_{BL} = 0.1287$.

Finally, the contribution of each group to I_V is also specified. From (Eq. 18) and (Eq. 19),

$$I_V = \frac{2\Delta_p}{2n^2 Y_m} = 2(C_{\Delta p,H} + C_{\Delta p,L})$$

$$C_{VH} = 2C_{\Delta p,H} = 0$$

$$C_{VL} = 2C_{\Delta p,L} = \frac{2(2 \cdot 140)}{2(8)^2(167.5)} = 0.0261$$ (Eq. 21)

Therefore $I_V = C_{VH} + C_{VL} = 0 + 0.02612 = 0.0261$ (Eq. 22)

Finally, the contribution of the each group to the overall Gini index is specified as;

Group H's contribution $C_H = C_{WH} + C_{BH} + C_{VH}$

Group L's contribution $C_L = C_{WL} + C_{BL} + C_{VL}$ (Eq. 23)

Therefore, for the numerical example here, the contribution of each income group to the total Gini index is;

$$\mathrm{C_H} = 0.1063433 \ + 0.14179 + 0 = 0.2481$$

$$\mathrm{C}_L = 0.024254 - 0.01306 + 0.02612 = 0.0373$$

So the overall Gini index is verified as:

$$\mathrm{I}_G = \mathrm{C_H} + \mathrm{C_L} = 0.2481333 + 0.0373141 = 0.2854$$

Therefore the contribution of each group to the overall Gini index is about 87% ($0.2481333 \div 0.2854474 = 0.8693$) from high income group and about 13% from the low income group.

I – 3 Dagum's Presentation of Gini Coefficient in Discrete Data

Deutsch and Silber (1999) presented the discrete version of the Dagum's (1997) decomposition of the Gini index. They also extended the methodology to develop the contribution of the each group to the ***changes*** in Gini coefficients. In this section, the expressions developed in the previous sections are utilized to demonstrate the discrete case of Dagum's decomposition (1997), which is equivalent to the decompositions of equations in (Eq. 14). It is first demonstrated for the discrete version of the Dagum's decomposition. Then it demonstrated for the changes of the Gini index and the contributions from each group to the changes in the next part.

Before exploring the Dagum's decomposition, several terms need to be defined and their interrelationships need to be organized. They are the Gini mean difference (GMD denoted by Δ_{HL}), the Gini ratio between group difference (GBG denoted by G_{HL}), the population shares (π_H for Group H and π_L for Group L), the income shares (σ_H for Group H and σ_L for Group L) and the relative economic affluence (REA denoted by D_{HL}).

The Gini mean difference (GMD) between the groups is the sum of the absolute difference between the incomes of the groups for the discrete data. The GMD for the two groups is defined as;

$$\Delta_{HL} = \frac{1}{n_H n_L} \sum_{S=1}^{n_H} \sum_{R=1}^{n_L} |Y_{HS} - Y_{LR}| \qquad \text{(Eq. 24)}$$

The equation (Eq. 24) expresses the average of the differences between members of the two groups. For the case of this numerical example here, $\Delta_{HL} = \frac{1}{n_H n_L} \sum_{S=1}^{n_H} \sum_{R=1}^{n_L} |Y_{HS} - Y_{LR}| = \frac{1660}{5(3)} = 110.67$. Next, focusing on the differences specified in Table 2, the Gini ratio between group (GBG, denoted by G_{HL}) is defined as follows;

$$G_{HL} = \frac{2 \sum_{S=1}^{n_H} \sum_{R=1}^{n_L} |Y_{HS} - Y_{LR}|}{2(n_H \cdot n_L)(Y_{mH} + Y_{mL})} = \frac{\Delta_{HL}}{(Y_{mH} + Y_{mL})} \qquad \text{(Eq. 25)}$$

Where Y_{mH} is the mean of the high income group, and Y_{mL} is the mean of the low income group.

To proceed with the analysis, the share of the population and income of each group are defined as;

$$\pi_H = \frac{n_H}{n} \quad \text{Population share of the high income group}$$
$$\pi_L = \frac{n_L}{n} \quad \text{Population share of the low income group} \qquad \text{(Eq. 26)}$$

$$\sigma_H = \frac{\sum_{S=1}^{n_H} Y_S}{\sum_{i=1}^{n} Y_i} \quad \text{Income share of the high income group}$$
$$\sigma_L = \frac{\sum_{R=1}^{n_L} Y_R}{\sum_{i=1}^{n} Y_i} \quad \text{Income share of the low income group} \qquad \text{(Eq. 27)}$$

In relating Δ_d and Δ_p to the share expressions the following terms are defined.

For $Y_{HS} > Y_{LR}$ $$d_{HL} = \left(\frac{1}{n_H n_L}\right)\sum_S \sum_R (Y_{HS} - Y_{LR}) \qquad \text{(Eq. 28)}$$

For $Y_{HS} < Y_{LR}$ $$p_{HL} = \left(\frac{1}{n_H n_L}\right)\sum_R \sum_S (Y_{LR} - Y_{HS}) \qquad \text{(Eq. 29)}$$

The (Eq. 28) defines the gross economic affluence (GEA) and the (Eq. 29) expresses the first-order moment of transvariation (FOMT).

Therefore, the value $\Delta_{HL} = d_{HL} + p_{HL}$ and $d_{HL} - p_{HL} = \frac{1}{n_H \cdot n_L}\left\{\sum_{S=1}^{n_H} \sum_{R=1}^{n_L} (Y_{HS} - Y_{LR})\right\}$. From (Eq.8) and (Eq. 9), then provide;

$$\Delta_d - \Delta_p = 2\left(\widetilde{\Delta_d} - \widetilde{\Delta_p}\right)$$
$$2\left[\left\{\sum_S \sum_R (Y_{HS} - Y_{LR}) \quad \text{for} \quad Y_{HS} > Y_{Lr}\right\} - \left\{\sum_S \sum_R (Y_{LR} - Y_{HS}) \quad \text{for} \quad Y_{HS} < Y_{Lr}\right\}\right] \qquad \text{(Eq. 30)}$$
$$= 2\left(\sum_S^{n_H} \sum_R^{n_L} (Y_{HS} - Y_{LR})\right)$$

Then $\sum_S^{n_H} \sum_R^{n_L} (Y_{HS} - Y_{LR}) = \frac{1}{2}\left(\Delta_d - \Delta_p\right)$. Substituting this expression into $d_{HL} - p_{HL}$, gives[5];

[5] Also, it can be shown that $d_{HL} - p_{HL} = (Y_{mH} - Y_{mL})$.
Proof:

$$d_{HL} - p_{HL} = \frac{1}{n_H . n_L}\left\{\sum_{S=1}^{n_H}\sum_{R=1}^{n_L}(Y_{HS} - Y_{LR})\right\} = \frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L} \qquad \text{(Eq. 31)}$$

Using (Eq. 24), (Eq. 28) and (Eq. 29), the relative economic affluence (REA, denoted by D_{HL}) is defined as[6]:

$$D_{HL} = \frac{d_{HL} - p_{HL}}{\Delta_{HL}} = \frac{\left(\frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L}\right)}{\Delta_{HL}} = \frac{\left(\frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L}\right)}{\left(\frac{1}{n_H n_L}\right)\sum_{R=1}^{n_H}\sum_{S=1}^{n_L}|Y_{HS} - Y_{LR}|} \qquad \text{(Eq. 32)}$$
$$= \frac{1}{2}\frac{(\Delta_d - \Delta_p)}{\sum_{S=1}^{n_H}\sum_{R=1}^{n_L}|Y_{HS} - Y_{LR}|}$$

In the case of further analysis, from (Eq. 25), (Eq.26) and (Eq. 27), yields;

$$G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H) = \frac{\Delta_{HL}}{Y_{mH} + Y_{mL}}\left(\frac{n_H}{n} \cdot \frac{\sum_{R=1}^{n_L} Y_R}{\sum_{i=1}^{n} Y_i} + \frac{n_L}{n} \cdot \frac{\sum_{S=1}^{n_H} Y_S}{\sum_{i=1}^{n} Y_i}\right)$$
$$= \frac{\Delta_{HL}}{Y_{mH} + Y_{mL}}\left\{\frac{n_H \cdot n_L(Y_{mH} + Y_{mL})}{n \cdot \sum_{i=1}^{n} Y_i}\right\} = \frac{n_H \cdot n_L \cdot \Delta_{HL}}{n^2 Y_m} \qquad \text{(Eq. 33)}$$

Also, (Eq. 24) implies,

$$\Delta_{HL} = \frac{1}{n_H n_L}\left\{\sum_{S=1}^{n_H}\sum_{R=1}^{n_L}|Y_{HS} - Y_{LR}|\right\}$$
$$= \frac{1}{n_H n_L}\left[\left\{\sum_S \sum_R (Y_{HS} - Y_{LR}) \quad for \quad Y_{HS} > Y_{Lr}\right\}\right. \qquad \text{(Eq. 34)}$$
$$\left. + \left\{\sum_S \sum_R (Y_{LR} - Y_{HS}) \quad for \ Y_{HS} < Y_{Lr}\right\}\right] = \frac{1}{n_H n_L}\left(\widetilde{\Delta_d} + \widetilde{\Delta_p}\right)$$

Now by using the notations defined above, the breakdown of the decomposition of the Gini concentration ratio expressed in (Eq. 14) may be expressed in terms of π_H,

---

$$d_{HL} - p_{HL} = \frac{1}{n_H . n_L}\sum_{S=1}^{n_H}\left(n_L \cdot Y_{HS} - \sum_{R=1}^{n_L} Y_{LR}\right) = \frac{1}{n_H . n_L}\sum_{S=1}^{n_H}(n_L \cdot Y_{HS} - n_L \cdot \overline{Y_{mL}})$$
$$= \frac{1}{n_H . n_L}\left\{n_L\left(\sum_{S=1}^{H} Y_{HS}\right) - n_H \cdot n_L \cdot \overline{Y_{mL}}\right\} = \frac{1}{n_H . n_L}(n_H \cdot n_L)(\overline{Y_{mH}} - \overline{Y_{mL}})$$
$$= (\overline{Y_{mH}} - \overline{Y_{mL}})$$

Further, from (Eq. 31) $\Delta_d - \Delta_p = 2n_H \cdot n_L(d_{HL} - p_{HL})$. Substituting this expression into the second equation in (Eq. 14),

$$I_B = \frac{(\Delta_d - \Delta_p)}{2n^2 Y_m} = \frac{2n_H n_L}{2n^2 Y_m}(d_{HL} - p_{HL}) = \frac{1}{Y_m}\pi_H \pi_L(Y_{mH} - Y_{mL}).$$

6 Also,

$$D_{HL} = \frac{d_{HL} - p_{HL}}{\Delta_{HL}} = \frac{\sum_{S=1}^{n_H}\sum_{R=1}^{n_L}(Y_{HS} - Y_{LR})}{\sum_{S=1}^{n_H}\sum_{R=1}^{n_L}|Y_{HS} - Y_{LR}|}$$

π_L,σ_H,σ_L,Δ_{HL},G_{HL},d_{HL},p_{HL} and D_{HL}. Unlike (Eq. 14) the between group inequality is explored first, and then the transvariational inequality. The within group inequality is explored at last.

First, the between-group-inequality is explored. From (Eq. 32) and (Eq. 33), it is shown that;

$$G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)D_{HL} = \left(\frac{n_H \cdot n_L \cdot \Delta_{HL}}{n^2 Y_m}\right)\left(\frac{\left(\frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L}\right)}{\Delta_{HL}}\right) = \frac{(\Delta_d - \Delta_p)}{2n^2 Y_m} = I_B \quad \text{(Eq. 35)}$$

Next, the transveriational inequality is developed from;

$$G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)(1 - D_{HL}) = \left(\frac{n_H \cdot n_L \cdot \Delta_{HL}}{n^2 Y_m}\right)\left(1 - \frac{\left(\frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L}\right)}{\Delta_{HL}}\right)$$

$$= \frac{(n_H n_L \Delta_{HL})\left(\Delta_{HL} - \frac{\Delta_d - \Delta_p}{2 \cdot n_H . n_L}\right)}{n^2 Y_m \Delta_{HL}} = \frac{n_H n_L \left(\frac{\Delta_{HL}(2 \cdot n_H . n_L) - (\Delta_d - \Delta_p)}{2 \cdot n_H . n_L}\right)}{n^2 Y_m} = \frac{\Delta_{HL}(2 \cdot n_H . n_L) - (\Delta_d - \Delta_p)}{2n^2 Y_m} \quad \text{(Eq. 36)}$$

From (Eq. 34),

$$\Delta_{HL}(2 \cdot n_H . n_L) = \left(\frac{1}{n_H n_L}\left(\widetilde{\Delta_d} + \widetilde{\Delta_p}\right)\right)(2 \cdot n_H . n_L) \quad \text{(Eq. 37)}$$

Substituting (Eq. 37) into (Eq. 36) and applying the relationship $\Delta_d = 2\widetilde{\Delta_d}$, $\Delta_P = 2\widetilde{\Delta_p}$. Thus (Eq. 35) implies the equivalence to I_V in (Eq. 14).

$$G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)(1 - D_{HL}) = \frac{2\left(\widetilde{\Delta_d} + \widetilde{\Delta_p}\right) - (\Delta_d - \Delta_p)}{2n^2 Y_m} = \frac{2\Delta_p}{2n^2 Y_m} = I_V \quad \text{(Eq. 38)}$$

Finally, the within group inequality is developed. Let the isolated Gini ratios of the high and low income group be stated as;

$$I_{WH} = \frac{\sum_{S=1}^{n_H} \sum_{S'=1}^{n_H} |Y_{HS} - Y_{HS'}|}{2n_H^2 Y_{mH}}$$
$$I_{WL} = \frac{\sum_{R=1}^{n_L} \sum_{R'=1}^{n_L} |Y_{LR} - Y_{LR'}|}{2n_L^2 Y_{mL}} \quad \text{(Eq. 39)}$$

Using $\mathrm{n}Y_m = \sum_{i=1}^{n} Y_i$, $n_H Y_{mH} = \sum_{S=1}^{n_H} Y_S$ and $\mathrm{n}_L \mathrm{Y}_{mL} = \sum_{R=1}^{n_L} Y_R$, (Eq. 26), (Eq. 27) and (Eq. 16),

$$\sigma_H\pi_H I_{WH} = \left(\frac{\sum_{S=1}^{n_H} Y_S}{\sum_{i=1}^{n} Y_i}\right)\left(\frac{n_H}{n}\right) I_{WH} = \left(\frac{n_H Y_{mH}}{nY_m}\right)\left(\frac{n_H}{n}\right) I_{WH}$$
$$= (\pi_H)^2 \left(\frac{Y_{mH}}{Y_m}\right) I_{WH}$$
$$= \left(\frac{n_H^2 Y_{mH}}{n^2 Y_m}\right)\left(\frac{\sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}|Y_{HS} - Y_{HS'}|}{2n_H^2 Y_{mH}}\right) \quad \text{(Eq. 40)}$$
$$= \frac{\sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}|Y_{HS} - Y_{HS'}|}{2n^2 Y_m} = C_{WH}$$

Similarly, $C_{WL} = \sigma_L\pi_L I_{WL}$, the within inequaility is;

$$C_{WH} + C_{WL} = \sigma_H\pi_H I_{WH} + \sigma_L\pi_L I_{WL} = I_w \quad \text{(Eq. 41)}$$

In sum, another presentation of (Eq. 14) is;

$$I_w = \sigma_H\pi_H I_{WH} + \sigma_L\pi_L I_{WL}$$
$$I_B = G_{HL}(\pi_H\cdot\sigma_L + \pi_L\cdot\sigma_H)D_{HL} \quad \text{(Eq. 42)}$$
$$I_V = G_{HL}(\pi_H\cdot\sigma_L + \pi_L\cdot\sigma_H)(1 - D_{HL})$$

These equations are components of the discrete estimates of the Dagum's decomposition[7].

For the simple numerical example presented in Part I, (Eq. 42), and the population and income shares are summarized in the following Table 3.

(Table 3 is shown on the next page.)

In this part, a simple numerical example is use to demonstrate how to compute the contribution of each group to the components of decomposition of Gini ratio. In addition, the discrete version of Dagum's presentation is demonstrated numerically. In the next part, the extension of the Dagum's presentation by Deutsch and Silber (1999) is used to show how the changes in each component of the decomposition can be expressed as a weighted sum of different sub-components of the Dagum's decomposition.

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7 From the first equation in (Eq. 42) and the footnote 4, the presentation of the Gini index in Milanovic (2005, p. 22) is verified. That is;

$$I_G = (I_{WH}\pi_H\sigma_H + I_W\pi_L\sigma_L) + \frac{1}{Y_m}(Y_{mH} - Y_{mL})\pi_H\pi_L + I_V$$

Table 3: Dagum (1997) Decomposition of the Sample Data in (Eq. 42)

Component in the decomposition	Value
Mean of the group	$Y_m = 167.5$
High mean group (Y_{mH})	$Y_{mH} = 202$
Low mean group (Y_{mL})	$Y_{mL} = 110$
Share of population	
High mean group (π_H)	$\pi_H = 0.625$
Low mean group (π_L)	$\pi_L = 0.375$
Share of income	
High mean group (σ_H)	$\sigma_H = 0.754$
Low mean group	$\sigma_L = 0.246$
Gini mean difference (GMD, Δ_{HL})	$\Delta_{HL} = 110.6667$
Gini between group differences (GBG, G_{HL})	$\mathrm{G}_{HL} = \dfrac{110.6667}{110+202} = 0.3547$
Gross economic affluence(GEA)	$d_{HL} = 101.333$
First order moment of transvariation (FOMT)	$p_{HL} = 9.333$
Relative economic affluence (REA)	$\mathrm{D}_{HL} = \dfrac{101.3333 - 9.33333}{110.6667} = 0.8313$

$$(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H) = (0.754 \cdot 0.375) + (0.246 \cdot 0.625) = 0.4366$$

$$I_{WH} = \frac{\sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}|Y_{HS} - Y_{HS'}|}{2(n_H)^2 Y_{mH}} = \frac{2280}{2 \cdot (25) \cdot 202} = 0.2257$$

$$I_{WL} = \frac{\sum_{R=1}^{n_L}\sum_{R'=1}^{n_L}|Y_{LR} - Y_{LR'}|}{2(n_L)^2 Y_{mL}} = \frac{520}{2 \cdot (9) \cdot 110} = 0.2626$$

Within group inequality	$I_W = \sigma_H \pi_H I_{WH} + \sigma_L \pi_L I_{WL}$ $= (0.753731 \cdot 0.625) \cdot 0.2257426$ $+ (0.246269 \cdot 0.375) \cdot 0.262626 = 0.1306$
Between group inequality	$I_B = \mathrm{G}_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)\mathrm{D}_{HL} = 0.354715 \cdot 0.437 \cdot 0.831325$ $= 0.1287$
Transvariaton inequality	$I_V = \mathrm{G}_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)(1 - D_{HL})$ $= 0.354715 \cdot 0.437(1 - 0.354715) = 0.0261$
Gini concentration ratio	$I_G = 0.2854$

Part II: Analysis of Change in Inequality by Deutsch and Silber (1999) and the Extended Example

Employing the expressions in (Eq. 35), (Eq. 38) and (Eq. 41), Deutsch and Silber (1999) described the method that leads to the analysis of the change in Gini index by the sub-components of I_W, I_B and I_V. To demonstrate this in the numerical example, a new set of observation for the high and low income group needs to be assumed. Then the change in Gini coefficients due to the change in contributions of each group is analyzed.

II – 1 The Presentation of the Decomposition in Multiplicative Form

When considering the income shares $\sigma_H = \pi_H \left(\frac{Y_{Hm}}{Y_m}\right), \sigma_L = \pi_L \left(\frac{Y_{Lm}}{Y_m}\right)$ and $\pi_L = 1 - \pi_H$, then the following three relationships can be established.

$$\begin{aligned} \pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H &= \pi_H \pi_L \left(\frac{Y_{mH} + Y_{mL}}{Y_m}\right) \\ &= \pi_H (1 - \pi_H) \left(\frac{Y_{mH} + Y_{mL}}{Y_m}\right) \end{aligned} \quad \text{(Eq. 43)}$$

$$G_{HL} D_{HL} = \left(\frac{\Delta_{HL}}{Y_{mH} + Y_{mL}}\right)\left(\frac{\mathrm{d_{HL}} - \mathrm{p_{HL}}}{\Delta_{\mathrm{HL}}}\right) = \frac{\mathrm{d_{HL}} - \mathrm{p_{HL}}}{Y_{mH} + Y_{mL}} \quad \text{(Eq. 44)}$$

and

$$\begin{aligned} G_{HL}(1 - D_{HL}) &= \left(\frac{\Delta_{HL}}{Y_{mH} + Y_{mL}}\right)\left(1 - \frac{\mathrm{d_{HL}} - \mathrm{p_{HL}}}{\Delta_{\mathrm{HL}}}\right) \\ &= \frac{\Delta_{HL} - (\mathrm{d_{HL}} - \mathrm{p_{HL}})}{Y_{mH} + Y_{mL}} = \frac{2p_{HL}}{Y_{mH} + Y_{mL}} \end{aligned} \quad \text{(Eq. 45)}$$

Also, for (Eq. 41)

$$\begin{aligned} \sigma_H \pi_H I_{WH} &= (\pi_H)^2 \left(\frac{Y_{mH}}{Y_m}\right) I_{WH} \\ \sigma_L \pi_L I_{WL} &= (\pi_L)^2 \left(\frac{Y_{mL}}{Y_m}\right) I_{WL} \end{aligned} \quad \text{(Eq. 46)}$$

Substituting (Eq. 46) into the first equation of (Eq. 42), (Eq. 43) and (Eq. 44) into the second equation of (Eq. 42), and (Eq. 43) and (Eq. 45) into the last equation of (Eq. 42), results in;

$$I_w = \sigma_H \pi_H I_{WH} + \sigma_L \pi_L I_{WL}$$
$$= (\pi_H)^2 \left(\frac{Y_{mH}}{Y_m}\right) I_{WH} + (1-\pi_H)^2 \left(\frac{Y_{mL}}{Y_m}\right) I_{WL}$$
$$I_B = G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H) D_{HL} = \pi_H(1-\pi_H)\left(\frac{d_{HL} - p_{HL}}{Y_m}\right) \qquad \text{(Eq. 47)}$$
$$I_V = G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)(1 - D_{HL}) = \pi_H(1-\pi_H)\left(\frac{2p_{HL}}{Y_m}\right)$$

In Deutsch and Silber (1999) the expressions in (Eq. 47) may also be expressed as functions of the relative economic affluence, D_{HL}, and the Gini ratio of between groups, G_{HL}. Let $\Pi = \pi_H(1-\pi_H)$, $M = \frac{Y_{mH}+Y_{mL}}{Y_m}$, and $\Gamma = \frac{2p_{HL}}{\Delta_{HL}}$, the equations in (Eq. 47) are restated as follows[8].

$$I_w = \sigma_H \pi_H I_{WH} + \sigma_L \pi_L I_{WL} = (\pi_H)^2 \left(\frac{Y_{mH}}{Y_m}\right) I_{WH} + (1-\pi_H)^2 \left(\frac{Y_{mL}}{Y_m}\right) I_{WL}$$
$$I_B = \pi_H(1-\pi_H)\left(\frac{d_{HL} - p_{HL}}{Y_m}\right)$$
$$= [\pi_H(1-\pi_H)]\left[\frac{Y_{mH}+Y_{mL}}{Y_m}\right]\left[\frac{\Delta_{HL}}{Y_{mH}+Y_{mL}}\right]\left[\frac{d_{HL}-p_{HL}}{\Delta_{HL}}\right] \qquad \text{(Eq. 48)}$$
$$= \Pi \cdot M \cdot G_{HL} \cdot D_{HL}$$
$$I_V = \pi_H(1-\pi_H)\left(\frac{2p_{HL}}{Y_m}\right) = [\pi_H(1-\pi_H)]\left[\frac{Y_{mH}+Y_{mL}}{Y_m}\right]\left[\frac{\Delta_{HL}}{Y_{mH}+Y_{mL}}\right]\left[\frac{2p_{HL}}{\Delta_{HL}}\right]$$
$$= \Pi \cdot M \cdot G_{HL} \cdot \Gamma$$

For the numerical example the values of the sub-components in (Eq.48) are computed and presented in Table 4.

[8] The term $\Gamma = (2p_{HL}/\Delta_{HL})$ is called ***income intensity of transvariation.***

Table 4:
Computed Values of the Sub-components of the (Eq. 48)
Presentation of the Decomposed Gini Coefficient

Components	Values
π_H^2	0.3906
π_L^2	0.1406
$\left(\frac{Y_{mH}}{Y_m}\right)$	1.2060
$\left(\frac{Y_{mL}}{Y_m}\right)$	0.6567
I_{WH}	0.2257
I_{WL}	0.2626
$\Pi = \pi_H \cdot \pi_L$	0.2344
$M = \frac{Y_{mH} + Y_{mL}}{Y_m}$	1.862
G_{HL}	0.3547
D_{HL}	0.8313
$\Gamma = \frac{2p_{HL}}{\Delta_{HL}}$	0.0843
Results for (Eq. 48)	
I_W	0.1306
I_B	0.1287
I_V	0.0261

The sub-components of the within group, between group and the transvariational inequality for (Eq. 48) are listed in the same manner as in Table. 3.

II – 2 The Extended Example and the Application of the Method by Deutsch and Silber (1999)

Resulting from the expressions in (Eq. 48), Deutsch and Silber (1999) further expressed the ***changes in*** the within group, between group, and the transvariational inequality as weighted functions of the sub-components of (Eq. 48). The changes are presented in the (Eq. 49) below, and the details of the weighted terms are shown in Table 5 where the difference operator is expressed by ∇.

Also, another artificially created data set of high and low income values is presented below. The data set is regarded as the second period data with respect to the Part I data regarded as the first period data. Using (Eq. 49), formulae in Table 5, and these two data sets, the changes in the parts of the decomposition of the Gini index are computed.

$$\begin{aligned}\nabla I_w &= \nabla(\sigma_H \pi_H I_{WH}) + \nabla(\sigma_L \pi_L I_{WL}) \\ &= \left\{\phi_{W1}[\nabla(\pi_H)^2] + \phi_{W2}\left[\nabla\left(\frac{Y_{mH}}{Y_m}\right)\right] + \phi_{W3}[\nabla I_{WH}]\right\} \\ &\quad + \left\{\phi_{W4}[\nabla(1-\pi_H)^2] + \phi_{W5}\left[\nabla\left(\frac{Y_{mL}}{Y_m}\right)\right] + \phi_{W6}[\nabla I_{WL}]\right\} \qquad \text{(Eq. 49)} \\ \nabla I_B &= \phi_{B1}[\nabla\Pi] + \phi_{B2}[\nabla M] + \phi_{B3}[\nabla G_{HL}] + \phi_{B4}[\nabla D_{HL}] \\ \nabla I_V &= \phi_{V1}[\nabla\Pi] + \phi_{V2}[\nabla M] + \phi_{V3}[\nabla G_{HL}] + \phi_{V4}[\nabla\Gamma]\end{aligned}$$

Notice that the weight terms are expressed as $\phi_{..}$'s; however the sum of these weights in each equation does not necessary add up to one.

The detailed expressions of the weight terms are summarized in Table 5, where the derivations for these terms are based on the analysis found in the Appendix of the Deutsch and Silber (1999).

(Table 5 is shown on the next page.)

To demonstrate the evaluation of the changes listed in Table 5, it is assumed that the following new set of data set is from another period. For the later period, indicated by #, the same income groups as those in Part I have other observed values;

Group $H^{\#}$={90, 140, 150, 180, 200, 230}

Group $L^{\#}$={60, 100, 200}

In this example, even the population sizes of the groups are assumed to have changed. The summary of these data for the computation of the Gini index is shown in Table 6.

Table 5: Detail of the Weight Terms in (Eq. 48) based upon the Appendix of Deutsch and Silber (1999).

	Changes subject to the weight		Weights in (Eq. 48)
Within Group	$\nabla(\pi_H)^2$	ϕ_{W1}	$\frac{\frac{Y_{mH}^1}{Y_m^1}\cdot I_{WH}^1+\frac{Y_{mH}^0}{Y_m^0}\cdot I_{WH}^0}{2}$
	$\nabla\left(\frac{Y_{mH}}{Y_m}\right)$	ϕ_{W2}	$\frac{([(\pi_H^1)^2+(\pi_H^0)^2])[I_{WH}^1+I_{WH}^0]}{4}$
	∇I_{WH}	ϕ_{W3}	$\frac{([(\pi_H^1)^2+(\pi_H^0)^2])\left[\frac{Y_{mH}^1}{Y_m^1}+\frac{Y_{mH}^0}{Y_m^0}\right]}{4}$
	$\nabla(1-\pi_H)^2$	ϕ_{W4}	$\frac{\frac{Y_{mL}^1}{Y_m^1}\cdot I_{WL}^1+\frac{Y_{mL}^0}{Y_m^0}\cdot I_{WL}^0}{2}$
	$\nabla\left(\frac{Y_{mL}}{Y_m}\right)$	ϕ_{W5}	$\frac{([(1-\pi_H^1)^2+(1-\pi_H^0)^2])[I_{WL}^1+I_{WL}^0]}{4}$
	∇I_{WH}	ϕ_{W6}	$\frac{([(1-\pi_H^1)^2+(1-\pi_H^0)^2])\left[\frac{Y_{mL}^1}{Y_m^1}+\frac{Y_{mL}^0}{Y_m^0}\right]}{4}$
Between Group	$\nabla\Pi$	ϕ_{B1} :	$\frac{(G_{HL}^1\cdot D_{HL}^1-G_{HL}^0\cdot D_{HL}^0)(M^1+M^0)}{4}$
	∇M	ϕ_{B2} :	$\frac{(G_{HL}^1\cdot D_{HL}^1+G_{HL}^0\cdot D_{HL}^0)(\Pi^1+\Pi^0)}{4}$
	∇G_{HL}	ϕ_{B3} :	$\frac{(\Pi^1\cdot M^1+\Pi^0\cdot M^0)(D_{HL}^1+D_{HL}^0)}{4}$
	∇D_{HL}	ϕ_{B4} :	$\frac{(\Pi^1\cdot M^1+\Pi^0\cdot M^0)(G_{HL}^1+G_{HL}^0)}{4}$
Transvariation	$\nabla\Pi$	ϕ_{V1} :	$\frac{(G_{HL}^1\cdot\Gamma^1-G_{HL}^0\cdot\Gamma^0)(M^1+M^0)}{4}$
	∇M	ϕ_{V2} :	$\frac{(G_{HL}^1\cdot\Gamma^1+G_{HL}^0\cdot\Gamma^0)(\Pi^1+\Pi^0)}{4}$
	∇G_{HL}	ϕ_{V3} :	$\frac{(\Pi^1\cdot M^1+\Pi^0\cdot M^0)(\Gamma^1+\Gamma^0)}{4}$
	$\nabla\Gamma$	ϕ_{V4} :	$\frac{(\Pi^1\cdot M^1+\Pi^0\cdot M^0)(G_{HL}^1+G_{HL}^0)}{4}$

The operator ∇ indicate the time difference of the variable. For example, using the superscript 0 and 1 for different time periods, $\nabla X = X^1 - X^0$.

Table 6: Summary of the H# ,L# data from Computing (Eq. 42)

Component in the decomposition	Value
Mean of the group	$Y_m^{\#} = 150$
High mean group (Y_{mH})	$Y_{mH}^{\#} = 165$
Low mean group (Y_{mL})	$Y_{mL} = 120$
Share of population	
High mean group (π_H)	π_H =0.6667
Low mean group (π_L)	π_L =0.3333
Share of income	
High mean group (σ_H)	σ_H =0.7333
Low mean group	σ_L =0.2667
Gini mean difference (GMD, Δ_{HL})	Δ_{HL} =72.7778
Gini between group differences (GBG, G_{HL})	$G_{HL} = \frac{72.7778}{165+120}$ =0.2554
Gross economic affluence(GEA)	d_{HL} =58.8889
First order moment of transvariation (FOMT)	p_{HL} =13.8889
Relative economic affluence (REA)	$D_{HL} = \frac{(58.8889-13.889)}{72.77777778}$ =0.618320611

$$(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H) = (0.667 \cdot 0.267) + (0.333 \cdot 0.733) = 0.422$$

$$I_{WH} = \frac{\sum_{S=1}^{n_H}\sum_{S'=1}^{n_H}\left|Y_{HS}-Y_{HS'}\right|}{2(n_H)^2 Y_{mH}} = \frac{1820}{2\cdot(36)\cdot 165} = 0.1532$$

$$I_{WL} = \frac{\sum_{R=1}^{n_L}\sum_{R'=1}^{n_L}\left|Y_{LR}-Y_{LR'}\right|}{2(n_L)^2 Y_{mL}} = \frac{560}{2\cdot(9)\cdot 120} = 0.2593$$

Within group inequality	$I_W = \sigma_H \pi_H I_{WH} + \sigma_L \pi_L I_{WL}$ =0.0979
Between group inequality	$I_B = G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H) D_{HL}$ =0.0667
Transvariaton inequality	$I_V = G_{HL}(\pi_H \cdot \sigma_L + \pi_L \cdot \sigma_H)(1 - D_{HL})$ =0.0412
Gini concentration ratio	I_G =0.2058

The sub-components of the equations in (Eq. 48) are presented in Table 7, along with the results shown in Table 4.

Table 7: The Values of the Sub-components of (Eq. 48) of the Decomposed Gini Index.

Variables that Changes in	# marked data (Period 1)	Part I data (Period 0) (Transferred from Table 4)
π_H^2	0.4444	0.3906
π_L^2	0.1111	0.1406
$\left(\frac{Y_{mH}}{Y_m}\right)$	1.1	1.2060
$\left(\frac{Y_{mL}}{Y_m}\right)$	0.8	0.6567
I_{WH}	0.1532	0.2257
I_{WL}	0.2593	0.2626
$\Pi = \pi_H \cdot \pi_L$	0.2222	0.2344
$\mathrm{M} = \frac{Y_{mH} + Y_{mL}}{Y_m}$	1.9	1.8627
G_{HL}	0.2554	0.3547
D_{HL}	0.6183	0.8313
$\Gamma = \frac{2p_{HL}}{\Delta_{HL}}$	0.3817	0.1687
Results of Decomposition Using the presentation in (Eq. 8)		
$\mathrm{I_W}$	0.0979	0.1306
$\mathrm{I_B}$	0.0667	0.1287
$\mathrm{I_V}$	0.0412	0.0261
$\mathrm{I_W}$	0.2058	0.2854

Using the values in Table 7, the weights are computed according to the formulas in Table 5. The resulting weights for each sub-components are presented in Table 8.

Table 8: The Values of the Weights for the Changes in (Eq. 48)

	Changes subject to the weight	Weight expressed in (Eq. 48)	
Within Group	$\nabla(\pi_H)^2$	$\phi_{W1} =$	0.2204
	$\nabla\left(\frac{Y_{mH}}{Y_m}\right)$	$\phi_{W2} =$	0.0791
	∇I_{WH}	$\phi_{W3} =$	0.4814
	$\nabla(1-\pi_H)^2$	$\phi_{W4} =$	0.2204
	$\nabla\left(\frac{Y_{mL}}{Y_m}\right)$	$\phi_{W5} =$	0.0328
	∇I_{WH}	$\phi_{W6} =$	0.0917
Sum of the weights		$\phi_{W1}+\phi_{W2}+\phi_{W3}+\phi_{W4}+\phi_{W5}+\phi_{W6} = 1.1258$	
Between Group	$\nabla\Pi$	$\phi_{B1} =$	-0.1289
	∇M	$\phi_{B2} =$	0.0517
	∇G_{HL}	$\phi_{B3} =$	0.3112
	∇D_{HL}	$\phi_{B4} =$	0.1310
Sum of the weights		$\phi_{B1}+\phi_{B2}+\phi_{B3}+\phi_{B4}$ =0.3650	
Transvariation	$\nabla\Pi$	$\phi_{V1} =$	0.0354
	∇M	$\phi_{V2} =$	0.0180
	∇G_{HL}	$\phi_{V3} =$	0.1182
	$\nabla\Gamma$	$\phi_{V4} =$	0.1310
Sum of the weights		$\phi_{V1}+\phi_{V2}+\phi_{V3}+\phi_{V4}$ =0.3025	

The operators ∇ indicate the time difference of the variables. For example, using the superscript 0 and 1 for different time periods $\nabla X = X^1 - X^0$.

So, even when the weights do not add up to one in each equation, the relative importance of the each sub-component can be seen. For example, the relative weight of the population squared to the between inequality can be $\phi_{W1}/(\phi_{W1}+\phi_{W2}+\phi_{W3}+\phi_{W4}+\phi_{W5}+\phi_{W6}) = \frac{0.220379}{1.125801} = 0.1958$. About 20% of the change in within group inequality is explained by the change in the square of the population of the high income group's income share.

II-3 Conclusion

This paper, by using the simple numerical examples, has demonstrated the the decomposition of Gini index described by Dagum (1997) and its further extensions by Deutsch and Silber (1999). Since this paper only presents the results from the artificially created data sets, the computed values cannot be used for any interpretation. However, the application of this methodology to real world data set is in order. The careful demonstration of the derivation process presented in this paper may help other researchers to visualize the overall process and to apply this methodology more easily.

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