A Method for Finding Core Allocations of Minimum Cost Forest Games

Masashi Umezawa

Abstract

In this paper, we study a method for finding core allocations of minimum cost forest games, where each supplier offers a different type of service to the customers and each customer wishes to be connected with the suppliers which he needs. We deal with problems of establishing a minimum cost network and of determining a fair cost allocation among customers. The characteristic function game is deduced from minimum costs for constructing subnetworks, which is called minimum cost forest game. We propose an algorithm for finding cost allocations in core when the games are satisfied sufficient conditions to have a nonempty core.

Keywords: Cooperative Game Thoery, Core, Cost Allocation, Minimum Cost Spanning Tree, Forest

1 Introduction

It is supposed that many new kinds of networks would emerge in the future. Then, we need enormous cost to construct information infrastructure or networks such as an intranetwork, a cable television network, and internet. Therefore, if many people or organizations share a network, it is significant to consider the fair cost allocation among the users.

For the problem of cost allocation on a network, Claus and Kleitman[2] consider a cost allocation in a minimum spanning tree, where there is exactly one supplier and all customers wish to be connected with the supplier. Bird[1] formulates this problem as a cooperative game which is called minimum cost spanning tree (MCST) game. Granot and Huberman[5] and Megiddo[7] prove that the MCST game has always a nonempty core.

On the other hand, for a model including more than one supplier, Rosenthal[9] deals with a model where all suppliers provide identical services, and customers need to maintain a path to at least one supplier, called minimum cost spanning forest game. Granot and Granot[4] consider fixed cost spanning forest (FCSF) game by introducing a fixed cost for establishing a supplier. Kuipers[6] introduces a generalized model of the MCST game, called minimum cost forest (MCF) game. Umezawa and Nishino[11] give sufficient conditions for the nonemptiness of the core of MCF game.

In an MCF game, customers and suppliers are represented by vertices on a network, and there may be more than one supplier. Services that suppliers provide are different. Each customer has a subset of suppliers which he needs, and he requires being connected with them directly or indirectly. Link cost is given for every pair of vertices. If some customers have common suppliers which they need, they might cooperatively construct a lower cost network by sharing links. We now consider the following problems. What is a minimum cost network such that all customers' demands are satisfied? Under what condition is there a fair cost allocation among the customers to establish the optimal network? Here, this problem is studied by a cooperative game theoretic approach, called MCF game.

Note that the MCF game does not always have a nonempty core. This paper provides a method for finding cost allocation included in the core when the sufficient conditions for the nonemptiness of the core proposed by Umezawa and Nishino[11] are satisfied. Given the value of the characteristic function, we can find a core allocation within a polynomial time by using the algorithm.

The remainder of this paper is organized as follows. In Section 2, the MCF game is formally described. In Section 3, we discuss the method of finding core allocations. In Section 4, we summarizes our findings and gives directions for future research.

2 MCF Game

Let us denote by $N=\{1, ..., n\}$be a set of customers and by $M = \{n+1, ..., n+m\}$ a set of suppliers. For each customer $i \in N$, $M(i) \subseteq$ M represents the set of suppliers required by i. It is assumed that $M(i) \neq \emptyset$ for all $i \in N$. For each $S \subseteq N$, $M(S) \equiv \cup_{i \in S} M(i)$. Let d be a nonnegative weight function on the edges in the complete graph with vertex set $N \cup M$. We denote by d_{ij} the cost of constructing a direct link between i and j. Assume that d_{ij} is symmetric, i. e., $d_{ij} = d_{ji}$. Given $E \subseteq P_2(N \cup M)$, the graph $G_E = (N \cup M, E)$ is defined, where $P_2(N \cup M)$ is the collection of all subsets of $(N \cup M)$ with cardinality two. We call G_E S-feasible if for any $i \in S \subseteq N$ and any $j \in M(i)$ there exists a path from i to j in G_E. For any $i \in S$, it is allowed that i uses the vertices of $N \cup M$ outside S and $M(S)$. Notice that N-feasible graph is always S-feasible for any $S \subseteq N$. Our objects are to find an N-feasible graph G_E such that the total link cost $\Sigma_{(i,j) \in E}\, d_{ij}$ is minimized and to allocate the cost of the optimal graph among the customers.

Note that a minimal N- or S-feasible graph may not be a tree. That is, a forest can be constructed if the cost of forest is lower than that of tree.

The cost allocation problem is analyzed by means of a cooperative game theory. The characteristic function of each coalition $S \subseteq N$ is defined as follows:

$$C(S) \equiv \min\{ \sum_{(i,j)\in E} d_{ij} \mid G_E \text{ is an } S\text{-feasible forest}\}.$$

It is clear that this characteristic function satisfies the subadditivity: $C(S) + C(T) \geq C(S \cup T)$, for any $S, T \subseteq N$.

We define a *minimum cost forest game* (MCF game) as an ordered pair (N, C). Moreover, core allocation is one of the popular solution concepts in the game theory, and is defined as follows:

$$\text{Core}(C) \equiv \{x \in R^n \mid \sum_{i\in N} x_i = C(N) \text{ and } \sum_{i\in S} x_i \leq C(S) \text{ for any } S \subset N\}.$$

Core gives fair allocations of the cost for the optimal N-feasible forest, if any.

3 Finding Core Allocations

We start with describing some definitions and lemmas needed for our argument.

Let us denote a *network* by an ordered pair (V, d) of the vertex set V and the weight function d of the edges of the complete graph with vertex set V .

Definition 3.1 Let (V, d) be a network and Γ be a minimum cost spanning tree on V. For each pair v, $w \in V$, let $d^{\Gamma}_{v,w}$ denote the maximum over all weights among the edges along the unique path from v to w in Γ, which we call *weight function*.

For a property of minimum cost spanning tree, see Ford and Fulkerson[3]. It has been shown that, given a network (V, d), the unique weight function d^{Γ} can be determined. For the proof, see Kuipers[6].

Lemma 3.1 ([6]) *Let (V, d) be a network, and Γ and Ω be two minimum spanning trees on V in this network. Let d^{Γ} and d^{Ω} be the weight functions defined in Definition 3.1. Then $d^{\Gamma} = d^{\Omega}$.*

For a network (V, d), the weight function d^{Γ} defined above is determined regardless of the particular minimum spanning tree Γ. In the following, we suppress Γ and denote the weight function by $\bar{d}$. We call $\bar{d}$ minimal weight function induced from the network (V, d), and the network $(V, \bar{d})$ minimal network. The minimal network has the following properties as verified in Kuipers[6].

Definition 3.2 An MCF game (N, C) is said to be *submodular* if, for every $S, T \subseteq N$,

$$C(S \cup T) + C(S \cap T) \leq C(S) + C(T).$$

For submodular game, see Shapley[10].

Lemma 3.2 ([6]) *Let $(N, \bar{C})$ be an MCF game with set of suppliers M, defined on a minimal network $(N \cup M, \bar{d})$. Then the game is submodular.*

The above Lemma implies that the MCF game on a minimal network always has a nonempty core.

We proceed to describe a notion of equivalence relation on a network. For any $i, i' \in N$, we say that i is related to i' by the binary relation $\approx$ and write $i \approx i'$ if $M(i) \cap M(i') \neq \emptyset$.

If there is a sequence $i_1, ..., i_k$ of the players such that $i_l \approx i_{l+1}$ for any $l = 1, ..., k-1$, we denote the extended relation by $i_1 \sim i_k$. It is clear that the relation $\sim$ defines an *equivalence relation*. Thus, N is always partitioned into the equivalence classes $N_1, ..., N_p$, where any pair of elements in each N_j ($j = 1, ..., p$) has the equivalence relation $\sim$. Accordingly, M is partitioned into $M_1, ..., M_p$. Let $V_j \equiv N_j \cup M_j$ for $j = 1, ..., p$. We call each V_j an *equivalence component* on a network.

By utilizing the concept of the equivalence relation, we proceed to argue about the MCF game. For any $U \subseteq (N \cup M)$, let $K(U)$ be the cost of a minimum spanning tree in the network (U, d). For each $j = 1, ..., p$, by using a minimum spanning tree Γ_j in the network (V_j, d) we can determine a unique minimal weight function $\tilde{d}$ and a minimal network $(V_j, \tilde{d})$ from Lemma 3.1. Hereafter, $\tilde{d}$ always refers to the weight function associated with the minimal network $(V_j, \tilde{d})$. For any $U_j \subseteq V_j$, we denote by $\tilde{K}(U_j)$ the cost of a minimum spanning tree in the network $(U_j, \tilde{d})$.

Now, consider the game $(N, \hat{C})$ defined on the network $(N \cup M, \hat{d})$, where $\hat{d}_{vw} = \tilde{d}_{vw}$ if $v,w \in Vj$ for some j and $\hat{d}_{vw} = d_{vw}$ otherwise.

We denote by $(N_j, \tilde{C})$ a game with set of customers N_j and set of suppliers M_j, defined on the network $(V_j, \tilde{d})$. Let $S_j \equiv S \cap N_j$ for $j = 1, ..., p$. The game $(N, \hat{C})$ has the following relationship with the game (N, C).

Lemma 3.3 ([11]) *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. Let $V_j = N_j \cup M_j$, $j = 1, ..., p$, be equivalence components. If*

$$C(N) = \sum_{j=1}^{p} K(V_j), \tag{1}$$

then $\hat{C}(S) = \sum_{j=1}^{p} \tilde{C}(S_j)$ for all $S \subseteq N$.

Theorem 3.1 ([11]) *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. Let $V_j = N_j \cup M_j$, $j = 1, ..., p$, be equivalence components. If*

$$C(N) = \sum_{j=1}^{p} K(V_j),$$

then the game (N, C) has a nonempty core.

Especially, it is proved that if an MCF game is composed of at most two equivalence classes, then it has a nonempty core.

Theorem 3.2 ([11]) *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. Suppose that N is composed of at most two equivalence classes of the players. Then, the game has a nonempty core.*

Theorem 3.1 and Theorem 3.2 assure us of the existence of core allocation. However, if we face the problem of allocating the cost for constructing a network, it is useful to provide the users with amounts of the cost allocation. Assuring the existence of the fair cost allocation is different from being able to give explicitly the allocation. Thus, we consider the algorithm for finding the cost allocation in core of the MCF game. It is obtained by utilizing the result in Shapley[10]. We consider the following two allocation rules under the assumption that the sufficient condition in Theorem 3.1, i.e., (1) holds.

Allocation Rule

Let π be a simple ordering of the players. We define $S^{\pi,k} = \{i \in N \mid \pi(i) \leq k\}$ for $k = 0, 1, ..., n$. Moreover, define

$$x_i^{\pi} = C(S^{\pi,\pi(i)}) - C(S^{\pi,\pi(i)-1}), \text{ for all } i \in N. \tag{2}$$

Then, each ordering π prescribes an allocation vector x^{π}. Note that the values of characteristic function C are interpreted as costs through this article. It is shown by Shapley[10] that if a game (N, C) with the set N of players and the characteristic function C is submodular, the allocation given by (2) is in the core. We consider applying this rule of the allocation to the MCF game.

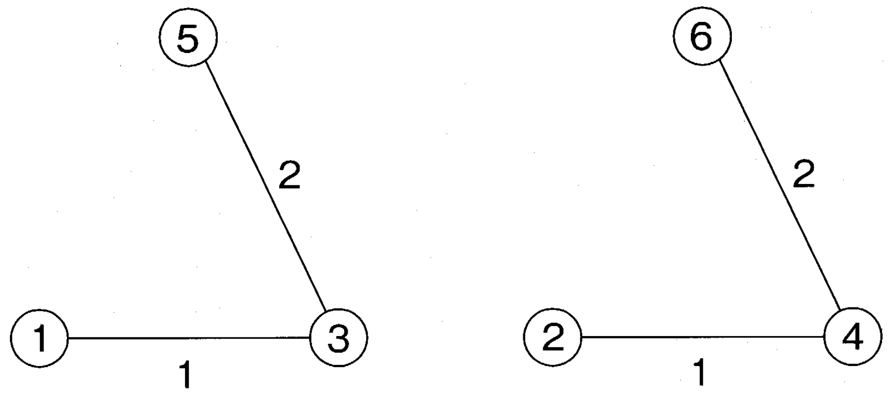
Umezawa and Nishino[11] show that the MCF game $(N_j ; \tilde{C})$ defined on the network $(V_j , \tilde{d})$ for each $j = 1, ..., p$, is submodular if (1) is satisfied. Moreover, it is verified that a core allocation of the MCF game (N,C) is able to be obtained from those of the MCF game $(N_j , \tilde{C})$ for $j = 1, ..., p$. Let x^j be an core allocation vector for the MCF game $(N_j , \tilde{C})$ for each $j = 1, ..., p$. Then, $x \equiv (x^1, ..., x^p)$ is in the core of the original MCF game (N,C).

For $i \in N_j$, where $j \in \{1, ..., p\}$, define $S_j^{\pi,k} = \{i \in N_j \mid \pi(i) \leq k\}$ for $k = 0, 1, ..., |N_j|$. Then, $S_j^{\pi,0} = \emptyset$ and $S_j^{\pi,|N_j|} = Nj$. Consider the following allocation:

$$x_i^{\pi} = \tilde{C}(S_j^{\pi,\pi(i)}) - \tilde{C}(S_j^{\pi,\pi(i)-1}). \tag{3}$$

If we let $x^j \equiv (x_i^{\pi})_{i \in N_j}$, then x^j is an core allocation vector for the MCF game $(N_j , \tilde{C})$. A vector $x \equiv (x^1, ..., x^p)$ is included in the core of the original MCF game (N, C). Thus, we can give a cost allocation

Figure 1: Minimum Cost Forest



in core of the MCF game.

Proposition 3.1 *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. If (1) is satisfied, then the cost allocation given by (3) for $i \in Nj$ is in the core of the game.*

Notice that a core allocation can be computed in a polynomial time by using this algorithm.

Example

Let N = {1, 2, 3, 4} be the set of players, and M = {5, 6} the set of suppliers. The sets of suppliers which each player requires are as follows: $M(1) = \{5\}$, $M(2) = \{6\}$, $M(3) = \{5\}$, and $M(4) = \{6\}$. Let $d_{13} = d_{24} = 1$, $d_{35} = d_{46} = 2$, $d_{15} = d_{26} = 3$, and the weight of all the other edges is 6. Then, the minimum cost forest is given by Figure 1. N is partitioned into the two equivalence classes N_1 and N_2, where $N_1 = \{1, 3\}$ and $N_2 = \{2, 4\}$. Accordingly, $M_1 = \{5\}$ and $M_2 = \{6\}$. The values of the weight function associated with the minimal network $(V_1, \tilde{d})$ are $\tilde{d}_{13} = 1$, $\tilde{d}_{35} = 2$, and $d_{15} = 2$. For the minimal network $(V_2, \tilde{d})$, $\tilde{d}_{24} = 1$, $\tilde{d}_{46} = 2$, and $\tilde{d}_{26} = 2$.

Let π^j be a simple ordering of the players in N_j for $j = 1, 2$. Assume that $\pi^1 = \{1, 3\}$ and $\pi^2 = \{2, 4\}$.

$$x_1^{\pi^1} = \tilde{C}(S_1^{\pi^1, \pi^1(1)}) - \tilde{C}(S_1^{\pi^1, \pi^1(1)-1}) = \tilde{C}(\{1\}) = 2,$$
$$x_3^{\pi^1} = \tilde{C}(S_1^{\pi^1, \pi^1(3)}) - \tilde{C}(S_1^{\pi^1, \pi^1(3)-1}) = \tilde{C}(\{1, 3\}) - \tilde{C}(\{1\}) = 3 - 2 = 1.$$

Similarly,

$$x_2^{\pi^2} = \tilde{C}(S_2^{\pi^2, \pi^2(2)}) - \tilde{C}(S_2^{\pi^2, \pi^2(2)-1}) = \tilde{C}(\{2\}) = 2,$$
$$x_4^{\pi^2} = \tilde{C}(S_2^{\pi^2, \pi^2(4)}) - \tilde{C}(S_2^{\pi^2, \pi^2(4)-1}) = \tilde{C}(\{2, 4\}) - \tilde{C}(\{2\}) = 3 - 2 = 1.$$

The cost allocation vector x = $(x_1^{\pi^1}, x_2^{\pi^2}, x_3^{\pi^1}, x_4^{\pi^2})$ is an element of the core of the MCF game (N, C).

4 Concluding Remarks

This paper studies a method for finding core allocations of minimum cost forest games, where each supplier offers a different type of service to the customers and each customer wishes to be connected with the suppliers which he needs. We present an algorithm for finding cost allocations in core when the games are satisfied sufficient conditions to have a nonempty core.

Each ordering π defines a cost allocation, so we may obtain the different cost allocation according to the ordering π. The study of the relation between the ordering and the allocation is of interest. Then, we can reconsider the cost allocation in terms of the fairness, though core allocation is one of the fair allocation.

The cost allocation vectors x obtained by the algorithm in this article are the vertices of the cores of the MCF games $(N_j, \tilde{C})$ $(j = 1, ..., p)$ defined on each minimal network $(V_j, \tilde{d})$. Notice that vertices are not generally near from the center of the core. We don't state the relation between x and the vertices of the core of the original MCF game (N, C). x^j is actually an inner point of the core for most of the MCF games (N_j, C). What vector in the core should be choose is one topic of research.

References

1) C. G. Bird: On cost allocation for spanning tree: A game theoretic approach. *Networks*, **6** (1976), 335–350.

2) A. Claus and D. J. Kleitman: Cost allocation for a spanning tree. *Networks*, **3** (1973), 289–304.

3) L. R. Ford and D. R. Fulkerson: *Flows in Networks*: (Princeton University Press, Princeton, 1962).

4) D. Granot and F. Granot: On the computational complexity of a cost allocation approach to a fixed cost spanning forest problem. *Mathematics of Operations Research*, **17** (1993), 765–780.

5) D. Granot and G. Huberman: Minimum cost spanning tree games. *Mathematical Programming*, **21** (1981), 1–18.

6) J. Kuipers: Minimum cost forest games. *International Journal of Game Theory*, **13** (1997), 367–377.

7) N. Megiddo: Cost allocation for steiner trees. *Networks*, **8** (1978), 1–6.

8) G. Owen: *Game Theory, Third Edition*: (Academic Press, San Diego, California, 1995).

9) E. C. Rosenthal: The minimum cost spanning forest game. *Economics Letters*, **23** (1987), 355–357.

10) L. S. Shapley: Core of convex games. *International Journal of Game Theory*, **1** (1971), 11–26.

11) M. Umezawa and H. Nishino: Sufficient Conditions for Nonempty Core of Minimum Cost Forest Games, *Journal of the Operations Research Society of Japan*, **46** (2003), no. 1, 35–43.